MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, PROBLEMS 3

Return by Monday 2nd Febraury

1. Let the polynomials $B_p(x)$ for $k \in \mathbb{N}$ be defined on $\mathbb{R}$ by $B_1(x) = x - \frac{1}{2}$, $B_{p+1}(x) = \int_0^x B_p(y)dy - \int_0^1 (1-y)B_p(y)dy$ and let $\hat{B}_p(k)$ denote the Fourier coefficient (relative to the family $e(kx)$ on $L^2(S_1)$).

(i) Prove that for $p > 1$ we have $B_p(1) = B_p(0)$ and deduce that B_p restricted to $\mathbb{R}/\mathbb{Z}$ is continuous.

(ii) Prove that $\hat{B}_P(0) = 0$ and $\hat{B}_p(k) = -(2\pi ik)^{-p}$ when $k \in \mathbb{Z}\backslash\{0\}$

(iii) Prove that when p is even, then $\sum_{k=1}^{\infty} k^{-p} = 2^{p-1}\pi^P(-1)^{p/2-1}B_p(0)$.

(iv) Prove that $B_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$, $B_4(x) = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 - \frac{1}{720}$ and that generally B_p is a polynomial of degree p with rational coefficients.

(v) Prove that $\zeta(2) = \frac{\pi^2}{6}$ and that $\zeta(4) = \frac{\pi^4}{90}$.

2.In the notation of the previous question, B_1 is the most interesting of the functions since it is not continuous on $\mathbb{R}/\mathbb{Z}$. It has a jump discontinuity at 0. Apart from the redefinition at 0 it coincides on $[0,1)$ with the sawtooth function $s(x)$ defined by $s(0) = 0$, $s(x) = x - \frac{1}{2}$ when $0 < x < 1$ and otherwise by periodicity with period 1. Let $E_K(x) = s(x) + \sum_{0<|k|\le K} \frac{e(kx)}{2\pi ik}$.

(i) Prove that $E_k(x)$ is an odd function of x.

(ii) Prove that if $x \notin \mathbb{Z}$, then $E_K'(x) = 1 + D_K(x)$ (the Dirichlet kernel).

(iii) Prove that if $0 < x < 1$, then $E_k(x) = E_k(x) - E_k(\frac{1}{2}) = \int_{\frac{1}{2}}^x D_K(y)dy$.

(iv) Prove that

$$\int_{\frac{1}{2}}^x D_K(y)dy = \left[\frac{1-\cos(\pi(2K+1)y)}{(2K+1)\pi\sin\pi y}\right]_{1/2}^x + \int_{1/2}^x \frac{1-\cos(\pi(2K+1)y)}{(2K+1)\sin^2\pi y}\cos\pi y dy.$$

(v) Prove that if $0 < x < 1$, then $|E_k(x)| \le \frac{2}{(2K+1)\pi\sin\pi x}$.

(vi) Prove that for all x, $|E_k(x)| \le \frac{1}{2}$. The facts that E_k is odd and that when $0 < x \le \frac{1}{2}$ we have $2x \le \sin\pi x \le \pi x$ are useful here.

(vii) Prove that $-\sum_{0<|k|\le K} \frac{e(kx)}{2\pi ik}$ converges to $s(x)$.

(viii) Prove that $\|E_K\|_2 = O(K^{-1/2})$ (relative to $L^2(\mathbb{R}/\mathbb{Z})$).