MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, PROBLEMS 1

Return by Wednesday 21st January

These exercises are essentially the same as in the text, so I have included a cross reference.

1. §1.1. Exercise 2. Prove that if $f_1, f_2 \ldots$ are real continuous functions of $\mathbb{R}$ and if for each $x \in \mathbb{R}$ we have $\lim_{n\to\infty} f_n(x)$ exists, then $\mathcal{A} = \{x : 0 \le f(x) < 1\}$ is measurable. Hint: Prove that $\mathcal{A} = \cup_{k\ge1} \cup_{m\ge1} \cap_{n\ge m} \{x : f_n(x) \le 1 - 1/k\}$.

2. §1.2. Exercise 2. Check that for fixed β, the inner product (α, β) is a continuous function of α.

3. §1.3. Exercise 9. For $x \in [0, 1]$ define the Haar function e_n^k by $e_0^0(x) = 1$ and, when $n \ge 0$, $1 \le k \le 2^n$, by

$$e_n^k(x) = \begin{cases} 2^{n/2} & \text{when } k-1 \le 2^n x < k - 0.5, \\ -2^{n/2} & \text{when } k - 0.5 \le 2^n x < k, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that they form a unit–perpendicular basis for $L^2[0, 1]$. Hint: One route to showing that they span is first to show that if f is perpendicular to them all, then $\int_0^x f = 0$ for all x of the form $k2^{-n}$, and deduce that $\int_{\mathcal{B}} f = 0$ for every measurable $\mathcal{B} \subset [0, 1]$.

4. §1.3. Exercise 14. Show that the family $\{f_n\}$ spans $L^2(Q)$ iff $(f, f_n) = 0$ for every n implies $f \equiv 0$. Hint: What is the annihilator of the family?