Factorization and Primality Testing Chapter 7 Pollard's Methods

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October 1, 2025

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- Then $P_{j+1} = P_{k+1}$ and so on.
- That is the sequence just repeats itself with period $k - j$.
- We can represent this as a ρ, where P_0 is at the base of the tail, and P_j is where the tail meets the loop.

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- There is no guarantee of finding one quickly, but sometimes one is found.
- The usual procedure is to stop after a certain amount of time and try a different polynomial f.

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- Thus sooner or later $y_j = y_k$ for some j, k with $j \neq k$.
- Then $x_j \equiv y_j \equiv y_k \equiv x_k \pmod{d}$. Probably, and hopefully, $x_j \neq x_k$ so $d | GCD(x_j - x_k, n)$ and the GCD will differ from n.

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- If we have $x_1, x_2, \ldots, x_s$ created at random, this is akin to the birthday paradox with a year that has p days and a class size of s.
- Thus we can expect that with s not much bigger than $\sqrt{p} < n^{1/4}$ we will find a solution.

Example 1

Let $n = 1133$ and $f(x) = x^2 + 1$. Of course $11|1133$.
Take $x_0 = 2$. Then $x_1 = 5$, $x_2 = 26$, $x_3 = 677$, $x_4 = 598$. Now

$$
\begin{aligned}
(x_1 - x_0, n) &= (3, 1133) = 1,\\
(x_2 - x_0, n) &= (24, 1133) = 1,\\
(x_3 - x_0, n) &= (675, 1133) = 1,\\
(x_4 - x_0, n) &= (596, 1133) = 1,\\
(x_2 - x_1, n) &= (21, 1133) = 1,\\
(x_3 - x_1, n) &= (672, 1133) = 1,\\
(x_4 - x_1, n) &= (593, 1133) = 1,\\
(x_3 - x_2, n) &= (651, 1133) = 1,\\
(x_4 - x_2, n) &= (572, 1133) = 11.
\end{aligned}
$$

Not very efficient, but it illustrates the idea.

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- Then $z_j = x_{2j}$, so we are computing x_j and x_{2j} simultaneously.
- If x_j and x_k with $j < k$ are the smallest pair with $x_j \equiv x_k \pmod{d}$, let $l = k - j$. Then $x_i \equiv x_{i+rl} \pmod{d}$ for every $i \geq j$ and every $r \geq 0$.

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- Take $i = l\lceil j/l \rceil$ so that $i \geq j$ and $r = \lceil j/l \rceil$.

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- Take $i = l\lceil j/l \rceil$ so that $i \geq j$ and $r = \lceil j/l \rceil$.
- Then $rl = i$ and so $x_i \equiv x_{2i} \pmod{d}$. Thus we only need check $GCD(x_{2i} - x_i, n)$ and this really speeds up the computations. In the previous example.

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Example 2

Let $n = 1133$, $f(x) = x^2 + 1$ and $x_0 = 2$.
Then we compute

$$x_1 = 5, x_2 = 26, \ (x_2 - x_1, n) = (21, 1133) = 1,$$
$$x_2 = 26, x_4 = 598, \ (x_4 - x_2, n) = (572, 1133) = 11.$$

That is more like it!

- A less obvious example

Example 3

Let $n = 713$, $f(x) = x^2 + 1$ and $x_0 = 2$.
Then we compute

$$x_1 = 5, x_2 = 26, \ (x_2 - x_1, n) = (21, 713) = 1,$$
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- There are a number of more sophisticated variants of this which are designed to speed the algorithm up.
- Generally there is no rigorous proof but it is believed that the run time is normally proportional to $\sqrt{p}$ where p is the smallest prime factor of n and so in the worst case, for a composite number the run time is proportional to $n^{1/4}$.

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- Obviously we do not want to compute and store $K!$, which will be huge.
- Thus for some a coprime with n we define $x_1 = a$ and successively compute

$$x_k \equiv x_{k-1}^k \pmod{n} \ \& \ GCD(x_k - 1, n) \quad (k = 2, 3, \ldots, K),$$

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- To explain the method we will assume a little more, namely that $p-1|K!$
- Obviously we do not want to compute and store $K!$, which will be huge.
- Thus for some a coprime with n we define $x_1 = a$ and successively compute

$$x_k \equiv x_{k-1}^k \pmod{n} \ \& \ GCD(x_k - 1, n) \quad (k = 2, 3, \ldots, K),$$

stopping if the GCD reveals a proper factor of n.
- Since n is large we can expect that $x_k \not\equiv 1 \pmod{n}$, but if $p|n$ and $p-1|k!$, so that $k! = m(p-1)$ for some m, then we have

$$x_k \equiv a^{k!} = (a^{p-1})^m \equiv 1 \pmod{p}.$$

- Consider our old friend 1133.

Example 4

Let $a = 2$. Thus $x_1 = 2, x_2 = 2^2 = 4, x_3 = 4^3 = 64,$

$$x_4 = 64^4 = 16777216 \equiv 719 \pmod{1133}, \quad (718, 1133) = 1,$$

$$x_5 = 719^5 = 192,151,797,699,599 \equiv 1101 \pmod{1133},$$

$$(1100, 1133) = 11.$$

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$$(1100, 1133) = 11.$$

- Now look at the less obvious example we considered above

Example 5

Let $n = 713$, & $a = 2$. Thus $x_1 = 2, x_2 = 2^2 = 4, x_3 = 4^3 = 64$,

$x_4 = 64^4 = 16777216 \equiv 326 \pmod{713}$, $(325, 713) = 1$, $x_5 =$

$326^5 = 3,682,035,745,376 \equiv 311 \pmod{713}, (310, 713) = 31$

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- The elliptic curve method is based on a similar basic idea but takes advantage of the richer underlying group structure of elliptic curves.