MATH 467 FACTORIZATION AND PRIMALITY TESTING, FALL 2024, PROBLEMS 12

Return by Monday 2nd December

1. Given $n \in \mathbb{N}$, let $R(n)$ denote the number of solutions of $n = x^2 - y^2$ in integers $x, y \in \mathbb{Z}$.
(i) Prove that if $n \in \mathbb{N}$ is an odd number, but not a perfect square, then $R(n) = 2d(n)$.
(ii) Prove that if $n \in \mathbb{N}$ is an even number but $4 \nmid n$, then $R(n) = 0$.
(iii) Prove that if $n \in \mathbb{N}$, $4|n$ but n is not a perfect square, then $R(n) = 2d(n/4)$.
(iv) Prove that if $n \in \mathbb{N}$ and n is an odd perfect square, then $R(n) = 2d(n)$.
(v) Prove that if $n \in \mathbb{N}$, $4|n$ and n a perfect square, then $R(n) = 2d(n/4)$.

2. (i) Prove that $(s^2 + at^2)(u^2 + av^2) = (su + atv)^2 + a(sv - tu)^2$. Deduce that if m and n are both of the form $x^2 + ay^2$, then so is mn.

3. (i) Prove that there is an arithmetic function f such that for every natural number n we have $\mu(n) = \sum_{m|n} f(m)$.
(ii) Prove that f is multiplicative, and give a formula for $f(p^k)$ when p is prime.

4. Show that
$$\left(\sum_{m|n} d(m)\right)^2 = \sum_{m|n} d(m)^3.$$

5. We define $\sigma(n)$ for $n \in \mathbb{N}$ to be the sum of the divisors of n,
$$\sigma(n) = \sum_{m|n} m.$$
(i) Prove that σ is a multiplicative function.
(ii) Evaluate $\sigma(1050)$.
(iii) Prove that
$$\sum_{m|n} \phi(m)\sigma(n/m) = nd(n).$$
(iv) Show that if $\sigma(n)$ is odd, then n is a square or twice a square.
(v) Prove that
$$\sum_{m|n} \mu(m)\sigma(n/m) = n.$$
(vi) Prove that
$$\sum_{m|n} \mu(n/m) \sum_{l|m} \mu(l)\sigma(m/l) = \phi(n).$$