Factorization and Primality Testing Chapter 8 The Quadratic Sieve

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- One of the lines of attack was through the use of continued fractions.
- It seems to have been periodically rediscovered, for example by Kraitchik and, most notably, by Lehmer and Powers in 1931 and then developed further by Morrison and Brillhart in 1975 who showed that the advent of modern computers made it a practical method.

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- and hopefully $GCD(A \pm R, n)$ provides a proper factor of n.

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- However SQUFOF (SQUareFOrmsFactorization) is sufficiently simple that it can be implemented on a pocket calculator and the instructor of this course has a version on his mobile phone.

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- In practice this in turn means Gaussian elimination.

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Definition 1

Given a positive real number B we say that an integer z is B-factorable when every prime factor p of z satisfies $p \le B$. To emphasise the fact that in our situation only certain primes (but also -1) may occur we will also use the term $\mathcal{P}$-factorable where $\mathcal{P}$ is a set of primes, probably augmented by -1.

- Note that the term B-smooth is commonly used instead. The word "smooth" has many better uses in mathematics.

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- **1.2.** *Set $p_0 = -1$, $p_1 = 2$ and find the odd primes $p_2 < p_3 < \ldots < p_K \leq B$ such that $\left(\frac{n}{p_k}\right)_L = 1$.*
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- **1.3.** *For $k = 2, \ldots, K$ find the solutions $\pm t_{p_k}$ to $x^2 \equiv n \pmod{p_k}$ by using* **(QC)**.

- **2. Sieving.**
  2.1. Let $N = \lceil \sqrt{n} \rceil$*. Sieve the sequence* $x^2 - n$ *with* $x = N + j$, $j = 0, \pm 1, \pm 2, \dots$ *until one has obtained a list of at least* $K + 2$ *B-factorable* $x_j^2 - n$ *and their factorizations (*$K + 2$ *is somewhat arbitrary and in the first example below is* $K + 1$*).*

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- *This could be done by using a matrix, with $K + 2$ rows so that the j–th column is a $K + 3$ dimensional vector in which the first entry is x_j, the second is $x_j^2 - n$, and the $k + 3$–rd entry is the exponent of p_k in $x_j^2 - n$.*

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- **2.2.** *For each prime p_k in $\mathcal{P}$ divide out all the prime factors p_k in each entry $x_j^2 - n$ with $x_j \equiv \pm t_{p_k} \pmod{p_k}$, recording the exponent in the $k + 3$-rd entry in the associated j-th vector. Once the primes start to grow this speeds things up significantly.*

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- **2.3.** *If the bottom entry in the j–th vector has reduced to 1, then $x_j^2 - n$ is B–factorable. If it has not completely factored then one can discard that column, or at least put it aside in case one needs to extend the factor base.*

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where $\mathbf{e}$ *is a* $K+2$ *dimensional vector of* 0*s and* 1*s (not all* 0*!).*
- *Note that the solution space may well be of dimension greater than* 1 *so then there would be multiple solutions.*

- **4. Factorization.**
  4.1. *Compute* $x = x_1^{e_1} x_2^{e_2} \dots x_{K+2}^{e_{K+2}}$ *modulo* n *and*

$$y = \sqrt{(x_1^2 - n)^{e_1}(x_2^2 - n)^{e_2} \dots (x_{K+2}^2 - n)^{e_{K+2}}}$$

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$$y = \sqrt{(x_1^2 - n)^{e_1}(x_2^2 - n)^{e_2} \dots (x_{K+2}^2 - n)^{e_{K+2}}}$$

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- **4.4.** *If necessary repeat for all solutions* **e** *until a non-trivial factor found.*

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- **5.3.** *Use another polynomial in place of* $x^2 - n$, *or rather, be a bit more cunning about the choice of the* x *in 2.1. Choose a large prime* p *for which* $b^2 - n \equiv 0 \pmod{p}$ *is soluble, and compute* b. *Then* $(px+b)^2 - n \equiv 0 \pmod{p}$ *and* x *can be chosen so that* $f(x) = ((px+b)^2 - n)/p$ *is comparatively small since* p *is large, so the sieving proceeds relatively speedily, there is a better chance of a complete factorization of* $f(x)$, *and we only have to augment the factor base with the prime* p.

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- The linear algebra can also be speeded up by various techniques, especially those developed for dealing with sparse matrices.
- Although the numbers in the following example are much smaller than would occur in a practice the example does illustrate the complexity of the basic quadratic sieve.

- **Example 8.1.** *Let* $n = 9487$ *and* $B = 30$.

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- Thus for each odd prime $p \le 30$ we need to ascertain whether n is a QR or a QNR modulo p.

$$\left(\frac{9487}{3}\right)_L = \left(\frac{1}{3}\right)_L = 1, \left(\frac{9487}{13}\right)_L = \left(\frac{10}{13}\right)_L = \left(\frac{36}{13}\right)_L = 1,$$
$$\left(\frac{9487}{5}\right)_L = \left(\frac{2}{5}\right)_L = -1, \left(\frac{9487}{17}\right)_L = \left(\frac{1}{17}\right) = 1,$$
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- Thus $\mathcal{P} = \{-1, 2, 3, 7, 11, 13, 17, 19, 29\}$.
- Then by bf (QC) $t_3 = \pm 1, t_7 = \pm 3, t_{11} = \pm 4,$

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- Now for a range of values of x near $\sqrt{n} \approx 97$ we factorise $f(x) = x^2 - n$. At this stage we throw away the x which do not completely factor in our factor base.

- Show Class467-08T1.pdf.

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- In the table above, in the column below each prime I have included the exponent of the prime which occurs in the factorisation and the residual factor after that prime has been factored out.

- I have included one such value, $x = 82$, below, so that you can see what happens. If n is proving awkward to factorise, one might go back and check to see if there are primes outside the factor base which occur in multiple places and then add them to the factor base. For example, $f(92)$ and $f(94)$ would completely factorise if we included the prime 31 in the factor base.

x	82	92	94
$f(x)$	-2763	-1023	-651
-1	2763,1	2763,0	651,1
2	2763,0	1023,1	651,0
3	307,2	341,1	217,1
7	307,0	341,0	31,1
11	307,0	31,0	31,0
13	307,0	31,0	31,0
17	307,0	31,0	31,0
19	307,0	31,0	31,0
29	307,0	31,0	31,0

- Let $\mathbf{v}(x)$ denote the vector of exponents in the factorization of $f(x)$, so that

$$\mathbf{v}(85) = \langle 1, 1, 1, 0, 0, 1, 0, 0, 1 \rangle,$$
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- Thus, modulo 9487,

$$85^2 \times 89^2 \times 98^2 \equiv (85^2-n)(89^2-n)(98^2-n)$$
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- Unfortunately

$$(741370 + 20358, 9487) = 1,$$
$$(741370 - 20358, 9487) = 9487.$$

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- 3. Whilst not apparent in the simple example above, we will need to work hard to find linear combinations of the vectors of exponents in which all the entries are even.
- This will involve some form of Gaussian elimination. The complexity is somewhat reduced by the fact that we only need to do this modulo 2, but it will still also require quite a lot of memory.

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- We can extract the exponents of each prime thus

$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 1 & 1 & 2 & 3 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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- Then we wish to find solutions to $\mathcal{M}\mathbf{e} \equiv \mathbf{0} \pmod 2$ other than $\mathbf{0}$.
- In other words we want the exponents in the prime factorisation of

$$f(x_1)^{e_1} \dots f(x_K)^{e_K}$$

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- The standard way of doing this is through Gaussian elimination, and it suffices to perform it modulo 2, although for the matrices which occur for large n, which are sparse there are faster methods. For the numbers used here Gauss' method will suffice.

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- On Class467-08T2.pdf I have listed the successive row operations, beginning with using the first row to eliminate the first entries in the other rows, and then using successive rows to eliminate the entries in the column corresponding to their leading entry.
- Here is the final form of the matrix, from which we can read off the equations for $\mathbf{e}$

$$\begin{pmatrix}1&0&0&0&0&0&0&1&0&0\\0&1&0&0&0&0&0&0&0&1\\0&0&1&0&0&0&1&0&0&0\\0&0&0&1&0&0&1&0&0&0\\0&0&0&0&1&0&0&1&0&0\\0&0&0&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix}$$

$$e_1+e_8\equiv 0 \pmod 2,\quad e_2+e_{10}\equiv 0 \pmod 2,$$
$$e_3+e_7\equiv 0 \pmod 2,\quad e_4+e_7\equiv 0 \pmod 2,$$
$$e_5+e_8\equiv 0 \pmod 2,\quad e_6+e_{10}\equiv 0 \pmod 2,$$
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- $$\begin{aligned} e_1 + e_8 &\equiv 0 \pmod 2, & e_2 + e_{10} &\equiv 0 \pmod 2,\\ e_3 + e_7 &\equiv 0 \pmod 2, & e_4 + e_7 &\equiv 0 \pmod 2,\\ e_5 + e_8 &\equiv 0 \pmod 2, & e_6 + e_{10} &\equiv 0 \pmod 2,\\ & & e_9 &\equiv 0 \pmod 2. \end{aligned}$$

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- Thus taking e_7, e_8 and e_{10} as the independent variables we see that

$$\big(f(x_3)f(x_4)f(x_7)\big)^{e_7}\big(f(x_1)f(x_5)f(x_8)\big)^{e_8}\times \big(f(x_2)f(x_6)f(x_{10})\big)^{e_{10}}$$

is always a perfect square.

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- The choices $e_7=1, e_8=e_{10}=0$ and $e_8=1, e_7=e_{10}=0$ correspond to the solutions used above.

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- The choices $e_7=1, e_8=e_{10}=0$ and $e_8=1, e_7=e_{10}=0$ correspond to the solutions used above.
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- **Example 8.3.** *Let* $n = 5479879$ *and take the sieving limit* $B = 50$*.*

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- *We have* $\sqrt{n} \approx 2340$. *For larger numbers such as n it is harder to obtain complete factorisations of* $f(x) = x^2 - n$.
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$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

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- We now apply Gaussian elimination and obtain

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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- Thus we find that

$$\begin{aligned} e_1 + e_4 &\equiv 0 \pmod 2, \\ e_2 + e_4 + e_5 &\equiv 0 \pmod 2, \\ e_3 + e_5 &\equiv 0 \pmod 2, \\ e_6 &\equiv 0 \pmod 2, \end{aligned}$$

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- and so each of

$$f(x_1)f(x_2)f(x_4),$$
$$f(x_2)f(x_3)f(x_5),$$

is a perfect square.

- Each of the following are squares.

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- We have

$$x_1 \times x_2 \times x_4 = 2198 \times 2225 \times 2373 = 11605275150$$

$$\begin{aligned} f(x_1)f(x_2)f(x_4) &= (-1)^2 \times 2^2 \times 3^{10} \times 5^6 \times 11^4 \times 31^2 \\ &= (2 \times 3^5 \times 5^3 \times 11^2 \times 31)^2 = 227873250^2 \end{aligned}$$

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- Thus

$$(11605275150 - 227873250, n)$$
$$= (11377401900, 5479879) = 5431$$

and

$$(1105275150 + 227873250, 5479879) = 1009.$$

- We can also check the second relationship.

$$x_2 \times x_3 \times x_5 = 2225 \times 2252 \times 2383 = 11940498100$$

$$\begin{aligned} f(x_2)f(x_3)f(x_5) &= (-1)^2 \times 2^2 \times 3^{12} \times 5^4 \times 11^4 \times 47^2 \\ &= (2 \times 3^6 \times 5^2 \times 11^2 \times 47)^2 = 207291150^2 \end{aligned}$$

Then

$$\begin{aligned} 11940498100 - 207291150 &= 11733206950, \\ 11940498100 + 207291150 &= 12147789250, \end{aligned}$$

$$(11733206950, 5479879) = 1009$$

and

$$(12147789250, 5479879) = 5431.$$

Note on Gaussian Elimination

- As part of the quadratic sieve we need to solve systems of linear congruences of the kind

$$\begin{aligned} a_{11}e_1 + a_{12}e_2 + \cdots + a_{1m}e_m &\equiv 0 \pmod{2}, \\ a_{21}e_1 + a_{22}e_2 + \cdots + a_{2m}e_m &\equiv 0 \pmod{2}, \\ &\vdots \qquad\qquad \vdots \\ a_{l1}e_1 + a_{l2}e_2 + \cdots + a_{lm}e_m &\equiv 0 \pmod{2}. \end{aligned}$$

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- For the numbers we will deal with Gaussian elimination is adequate, and has the merit of being straightforward.

Note on Gaussian Elimination

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- We can write this more succinctly in matrix notation as

$$\mathcal{A}\mathbf{e} = \mathbf{0}$$

where

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{l1} & a_{l2} & \cdots & a_{lm} \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

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- The first observation that can be made is that it is immaterial as to the order in which we write the equations so at any state we can interchange them if it is convenient to do so. Thus we can suppose initially that a left-most non-zero entry is in the top row. This is sometimes called a *pivot*.

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{l1} & a_{l2} & \cdots & a_{lm} \end{pmatrix}$$

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- Our second observation is that in our original system of linear congruences we can take one equation and subtract it from another. This is equivalent to taking the corresponding row in the matrix and subtracting it from the second corresponding row.

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{l1} & a_{l2} & \cdots & a_{lm} \end{pmatrix}$$

- When Gaussian elimination is applied generally in the real world one can even take real multiples of one row from another, but in this world we have the much simple environment of having only zeros and ones. Note that if subtraction gives -1 this is the same as 1.

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- Denote the pivot in the top row by a_{j1}. We now take the first row and subtract it from every row with $a_{jk} = 1$. Thus the new matrix will have $a_{j1} = 1$ and all the entries to the left and below it are 0.

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- We now repeat this process with the submatrix formed from the rows $j + 1$ through m.

- We continue in this way until we have reduced the matrix to *echelon* form

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} \\ 0 & 1 & a_{23} & a_{24} & \cdots & a_{2m} \\ 0 & 0 & 0 & 1 & \cdots & a_{3m} \\ 0 & 0 & 0 & 0 & \cdots & \vdots \\ & & \vdots & & & \vdots \end{pmatrix}.$$

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- Note that the matrix might well have zeros on the diagonal from some point on. If so some of the rows at the bottom of the matrix are likely to consist of all zeros.

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- Note that the matrix might well have zeros on the diagonal from some point on. If so some of the rows at the bottom of the matrix are likely to consist of all zeros.
- The first 1 in a row is called a *pivot*.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} \\ 0 & 1 & a_{23} & a_{24} & \cdots & a_{2m} \\ 0 & 0 & 0 & 1 & \cdots & a_{3m} \\ 0 & 0 & 0 & 0 & \cdots & \vdots \\ & & \vdots & & & \vdots \end{pmatrix}.$$

- Starting from the bottom of the matrix we now use these pivots to remove any non-zero entry above the pivot.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} \\ 0 & 1 & a_{23} & a_{24} & \cdots & a_{2m} \\ 0 & 0 & 0 & 1 & \cdots & a_{3m} \\ 0 & 0 & 0 & 0 & \cdots & \vdots \\ & & \vdots & & & \vdots \end{pmatrix}.$$

- Starting from the bottom of the matrix we now use these pivots to remove any non-zero entry above the pivot.
- Thus the last matrix would take on the shape

$$\begin{pmatrix} 1 & 0 & a_{13} & 0 & \cdots & a_{1m} \\ 0 & 1 & a_{23} & 0 & \cdots & a_{2m} \\ 0 & 0 & 0 & 1 & \cdots & a_{3m} \\ 0 & 0 & 0 & 0 & \cdots & \vdots \\ & & \vdots & & & \vdots \end{pmatrix}.$$

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} \\ 0 & 1 & a_{23} & a_{24} & \cdots & a_{2m} \\ 0 & 0 & 0 & 1 & \cdots & a_{3m} \\ 0 & 0 & 0 & 0 & \cdots & \vdots \\ & & \vdots & & & \vdots \end{pmatrix}.$$

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- This is called *reduced echelon* form.

$$\begin{pmatrix} 1 & 0 & a_{13} & 0 & \cdots & a_{1m} \\ 0 & 1 & a_{23} & 0 & \cdots & a_{2m} \\ 0 & 0 & 0 & 1 & \cdots & a_{3m} \\ 0 & 0 & 0 & 0 & \cdots & \vdots \\ & & \vdots & & & \vdots \end{pmatrix}.$$

- The variables corresponding to pivots are the dependent variables and the other variables are the independent ones. The values for the dependent variables are then easily read off in terms of the independent ones.

- Thus in Example 8.1 the reduced echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

e_1, e_2, e_3, e_4, e_5, e_6 and e_9 are dependent variables and the e_7, e_8 and e_{10} can be chosen at random.