Factorization and Primality Testing Chapter 6 Primality and Probability

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Miller-Rabin

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- In its simplest form the Miller-Rabin test is a test for composites, although with some compromises it is also an effective test for primality.
- The basic question is how easy is it to find a witness a in the following theorem when n is composite and how easy is it to determine that there is no witness when n is prime?

Theorem 1

Let $n \in \mathbb{N}$ be odd, $n > 1$ and take out the powers of 2 from $n-1$ so that

$$n - 1 = 2^u v$$

where v is odd. Choose $a \in \{2, 3, \ldots, n-2\}$. If

$$a^v \not\equiv 1 \pmod{n} \text{ and } a^{2^w v} \not\equiv -1 \pmod{n} \text{ for } 1 \le w \le u-1, \tag{1.1}$$

then n is composite and a is a **witness**.

- **Theorem 1.** Let $2 \nmid n \in \mathbb{N}$, $n > 1$ and suppose $n - 1 = 2^u v$ and $2 \nmid v$. Choose $a \in \{2, 3, \ldots, n-2\}$. If $a^v \not\equiv 1 \pmod{n}$ and

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- **Proof.** The proof of the theorem is quite simple.
- If $(a, n) > 1$, then (1.1) will hold and n will be composite. Suppose that $(a, n) = 1$ and n were to be prime. Then by Fermat-Euler we have $n | a^{n-1} - 1 =$

$$a^{2^u v} - 1 = (a^v - 1)(a^v + 1)(a^{2v} + 1) \ldots (a^{2^{u-1} v} + 1) \quad (1.2)$$

and n would have to divide one of the factors on the right, contradicting the hypothesis.

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- It is useful to eliminate some easily checked possibilities.
- A. Check n for small prime factors p for, say, $p \le \log n$.
- B. Check that n is not a prime power, $n = p^k$. One can do this by checking to see if

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for $2 \le k \le \frac{\log n}{\log 2}$.
- Now if n is composite it will have to have two different prime factors.

- The next theorem tells us what is happening when n has at least two different prime factors.

Theorem 2

If n is odd and has at least two different prime factors p and q, then they can be chosen so that

$$p-1=2^j l,\ q-1=2^k m, j\le k,$$

and then there are a with $(a,n)=1$ and

$$\left(1+\left(\frac{a}{p}\right)_L\right)\left(1-\left(\frac{a}{q}\right)_L\right)>0$$

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- In other words in this case witnesses to compositeness certainly exist.

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- As it stands this theorem only proves the existence of witnesses.
- Since we do not expect to have found numerical values for p or q, it does not tell us how to find the a.
- However it can be used to show that we do not have to search very far.

- Consider: a is a witness when $(a, n) = 1$ and

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$$\left(1+\left(\frac{a}{p}\right)_L\right)\left(1-\left(\frac{a}{q}\right)_L\right)>0.$$
- When $(a,n)=1$, $\frac{1}{4}\left(1+\left(\frac{a}{p}\right)_L\right)\left(1-\left(\frac{a}{q}\right)_L\right)$ is 0 or 1, and when it is 1, a is a witness.

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- Thus the number of witnesses for n is at least

$$\sum_{\substack{a=1\\(a,n)=1}}^{\phi(n)}\frac14\left(1+\left(\frac{a}{p}\right)_L\right)\left(1-\left(\frac{a}{q}\right)_L\right).$$

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$$\sum_{\substack{a=1\\(a,n)=1}}^{\phi(n)}\left(\frac{a}{p}\right)_L=\sum_{\substack{a=1\\(a,n)=1}}^{\phi(n)}\left(\frac{a}{q}\right)_L=\sum_{\substack{a=1\\(a,n)=1}}^{\phi(n)}\left(\frac{a}{pq}\right)_J=0.$$

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- Hence $\displaystyle\sum_{\substack{a=1\\(a,n)=1}}^{\phi(n)} \frac{1}{4}\left(1+\left(\frac{a}{p}\right)_L\right)\left(1-\left(\frac{a}{q}\right)_L\right)=\frac{\phi(n)}{4}.$

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- Then the probability that none of them are witnesses is at most $(3/4)^N$.

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- Therefore at least a quarter of all reduced residues modulo n act as witness.
- Hence we can proceed by picking N values of a at random.
- Then the probability that none of them are witnesses is at most $(3/4)^N$.
- Therefore if we pick, say, at least $10\log n$ numbers a at random, then we can be practically certain of finding a witness.

- If we want some kind of absolute certainty, then we can assume the truth of the Riemann hypothesis for the three functions $L(s;\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}$ with

$$\chi(m)=\left(\frac{m}{p}\right)_L,\ \chi(m)=\left(\frac{m}{q}\right)_L,\ \chi(m)=\left(\frac{m}{pq}\right)_J,$$

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- This hypothesis implies that for $N=2(\log n)^2$ we have

$$\sum_{\substack{r\le N\\ r \text{ prime}}}\left(1-\frac{r}{N}\right)\left(1+\left(\frac{r}{p}\right)_L\right)\left(1-\left(\frac{r}{q}\right)_L\right)\log r>0.$$

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- In turn, this tells us that not only is there a witness $a \le 2(\log n)^2$, but we can suppose that it is prime.

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- In turn, this tells us that not only is there a witness $a\le 2(\log n)^2$, but we can suppose that it is prime.
- There is even some belief that one does not have to search beyond $C(\log n)\log\log n$.

- **Theorem 2.** If n is odd and has at least two different prime factors p and q, then they can be chosen so that $p-1=2^j l$, $q-1=2^k m, j \leq k$, and then there are a with $(a,n)=1$ and $\left(1+\left(\frac{a}{p}\right)_L\right)\left(1-\left(\frac{a}{q}\right)_L\right)>0$ and such an a is a witness.

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- We need to prove this theorem.

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- We need to prove this theorem.
- **Proof.** Let p and q be as given. If we choose a QR x modulo p and a QNR y modulo q, then by the Chinese Remainder Theorem, it follows that there are a with $a\equiv x \pmod p$, $\equiv y \pmod q$ and $(a,n)=1$ so that a satisfies the hypothesis. Recall from Theorem 1 that u and v are defined by $n-1=2^u v$ where v is odd. If $a^{n-1}\not\equiv 1 \pmod n$, then none of the factors on the right of

$$a^{n-1}-1=a^{2^u v}-1=(a^v-1)(a^v+1)(a^{2v}+1)\dots(a^{2^{u-1}v}+1)$$

can be divisible by n, so any such a will be a witness. Thus we can suppose that we have $a^{n-1}\equiv 1 \pmod n$.

- Recall $n-1=2^u v$ where v is odd and

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- For $0 \le w \le u-1$ we have

$$a^{2^w v} + 1 = (a^v - 1 + 1)^{2^v} + 1 \equiv 2 \pmod{a^v - 1}.$$

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- Hence $\left(a^v-1, a^{2^w v}+1\right)|2$.

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- Likewise when $0\le w<x\le u-1$

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- Hence $\left(a^v-1, a^{2^w v}+1\right)|2$.
- Likewise when $0\le w<x\le u-1$
$$a^{2^x v}+1=(2^{2^w v}+1)^{2^{x-w}}+1\equiv 2 \pmod{a^{2^w v}+1}$$
- Therefore $\left(a^{2^w v}+1, a^{2^x v}+1\right)|2$.

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- Likewise when $0 \le w < x \le u-1$

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- Therefore $\left(a^{2^w v}+1, a^{2^x v}+1\right)|2$.
- Thus p and q, and *a fortiori* n cannot divide two different factors.

- The hypothesis implies that

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- In particular $0 \le i < j \le k$.

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- Thus the best bound for a leads to questions which have a similar provenance to that concerning the least quadratic non-residue $n_2(p)$ discussed in Chapter 5.
- In particular Linnik's work quoted there suggests that any composite n with no small witnesses would be incredibly rare.

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- 6. If no witness a found with $a \le \min\left\{2(\log n)^2, n-2\right\}$, then declare that n is prime.
- There are one further wrinkle that can be tried. Before doing the divisibility checks in 4, check that $(a,n)=1$ (or $a \nmid n$ if a is prime) because otherwise one has a proper divisor of n and not only is n composite but one has found a factor.

- A trivial but illustrative

Example 3

Let $n = 133$. Then

$$n - 1 = 2^2 \times 33$$

and

$$2^{33} \equiv 50 \pmod{133},\ 2^{66} \equiv 106 \pmod{133}$$

so

$$n \nmid 2^{33} - 1,\ n \nmid 2^{33} + 1,\ n \nmid 3^{66} + 1$$

Thus n is composite and a is a witness.

- Primality in a non-trivial case is best left to a computer program. But to illustrate the method here is an example.

Example 4

Let $n = 11$. Then $n - 1 = 2 \times 5$ and we have the following

$$\begin{aligned} 2^5 &= 32 \equiv -1 \pmod{11}, & 3^5 &= 243 \equiv 1 \pmod{11} \\ 4^5 &\equiv (2^5)^2 \equiv 1 \pmod{11}, & 5^5 &= 3125 \equiv 1 \pmod{11} \\ 6^5 &= (-5)^5 \equiv -1 \pmod{11}, & 7^5 &= (-4)^5 \equiv -1 \pmod{11} \\ 8^5 &= (-3)^5 \equiv -1 \pmod{11}, & 9^5 &= (3^5)^2 \equiv 1 \pmod{11} \end{aligned}$$

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- Even for a number like 211 this would be heavy handed and is one of the reasons for an initial range of trial division. For large n one will only need to consider a relatively small range of a.

- We have already used the term "probabilistic" informally in the previous section without saying precisely what we mean.

Definition 5

Suppose that we have a finite set $\mathcal{A}$ of cardinality M, and a subset $\mathcal{B}$ of cardinality N. In general we will suppose that the elements of $\mathcal{B}$ have some special property that marks them out from those in the complement of $\mathcal{B}$ with respect to $\mathcal{A}$. If we pick an element of $a \in \mathcal{A}$ without fear or favour, then we define the probability that $a \in \mathcal{B}$ as

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- Fortunately we have no need of that here.

- This comes up frequently

Example 6

Let $\mathcal{A} = \{1, 2, \ldots, M\}$, let $q \in \mathbb{N}$ and $0 \le r < 1$ and let

$$\mathcal{B}(q, r) = \{a \in \mathcal{A} : a \equiv r \pmod{q}\}.$$

Then $$N = \operatorname{card} \mathcal{B}(q, r) = 1 + \left\lfloor \frac{M - r}{q} \right\rfloor.$$

Now $$\frac{M - r}{q} - 1 < \left\lfloor \frac{M - r}{q} \right\rfloor \le \frac{M - r}{q}$$

and so $$-1 < -\frac{r}{q} < N - \frac{M}{q} \le 1 - \frac{r}{q} < 1.$$

Therefore $$-\frac{1}{M} + \frac{1}{q} < \frac{N}{M} < \frac{1}{q} + \frac{1}{M}.$$

Thus if M is large compared with q, then we can see that the probability that an element of a is in $\mathcal{B}$ is close to $\frac{1}{q}$.

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- The fallacy here is that we are dealing with more than just pairs.

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- Thus the probability that at least two members of the class share a birthday is

$$1-\rho(s) = 1-\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\dots\left(1-\frac{s-1}{365}\right).$$

s	$\rho(s)$	s	$\rho(s)$
21	.5563 ...	22	.5243 ...
23	.4927 ...	24	.4616 ...
25	.4313 ...	26	.4017 ...
27	.3731 ...	28	.3455 ...
29	.3190 ...	30	.2936 ...
31	.2695 ...	32	.2466 ...
33	.2250 ...	34	.2046 ...
35	.1856 ...	36	.1678 ...
37	.1512 ...	38	.1359 ...
39	.1217 ...	40	.1087 ...
41	.0968 ...	42	.0859 ...
43	.0760 ...	44	.0671 ...
45	.0590 ...	46	.0517 ...
47	.0452 ...	48	.0394 ...
49	.0342 ...	50	.0296 ...

  The probability $\rho(s)$ that a class of size s has no two birthdays the same.

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- We need to generalize this.
- Let D be the number of possible values for each entry in the s-tuple - so we are now supposing that our year has D days!
- Then $M = \operatorname{card} A = D^s$ and $N = \operatorname{card} B$ is

$$N = D(D-1)\dots(D-N+1)$$

so that the probability that there are no coincidences in the entries in an arbitrary s-tuple is

$$\frac{N}{M} = \left(1-\frac{1}{D}\right)\left(1-\frac{2}{D}\right)\dots\left(1-\frac{s-1}{D}\right).$$

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$$\rho(s) = \prod_{k=1}^{s-1}\left(1 - \frac{k}{D}\right) < \sigma.$$
- Since it is easier to work with sums than products, we can rewrite this as
$$\log\frac{1}{\rho(s)} = \sum_{k=1}^{s-1}\log\frac{1}{1-\frac{k}{D}} > \log\frac{1}{\sigma}.$$

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- and since all the terms are positive we have

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- $$\log \frac{1}{\rho(s)} = \sum_{k=1}^{s-1} \log \frac{1}{1-\frac{k}{D}}.$$
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- In other words, if s is large compared with $\sqrt{D}$, then it will be almost certain that there will be coincidences.
- By the way, some attacks on security systems take advantage of this and we will make use of it later in one of the factoring attacks.