MATH 467 FACTORIZATION AND PRIMALITY TESTING, FALL 2023, PROBLEMS 11

Return by Monday 4th December

1. (i) Prove that there is an arithmetic function f such that for every natural number n we have $\mu(n) = \sum_{m|n} f(m)$.

(ii) Prove that f multiplicative, and give a formula for $f(p^k)$ when p is prime.

2. Give an example of a totally multiplicative function f for which $\sum_{m|n} f(m)$ is not totally multiplicative.

3. Show that

$$\left(\sum_{m|n} d(m)\right)^2 = \sum_{m|n} d(m)^3.$$

4. We define $\sigma(n)$ for $n \in \mathbb{N}$ to be the sum of the divisors of n,

$$\sigma(n) = \sum_{m|n} m.$$

(i) Prove that σ is a multiplicative function.
(ii) Evaluate $\sigma(1050)$.
(iii) Prove that

$$\sum_{m|n} \phi(m)\sigma(n/m) = nd(n).$$

(iv) Show that if $\sigma(n)$ is odd, then n is a square or twice a square.
(v) Prove that

$$\sum_{m|n} \mu(m)\sigma(n/m) = n.$$

(vi) Prove that

$$\sum_{m|n} \mu(n/m) \sum_{l|m} \mu(l)\sigma(m/l) = \phi(n).$$

5. (i) Show that every odd number n can be written as the difference of two squares, $n = x^2 - y^2$. How many different choices for the integers x and y are there?

(ii) Prove that $(u^2 - v^2)(x^2 - y^2) = (ux + vy)^2 - (uy + vx)^2$ Deduce that if m and n are both the difference of two squares, then so is mn.