MATH 467 FACTORIZATION AND PRIMALITY, FALL TERM 2023, PROBLEMS 1

Divisibility

Return by Monday 28th August

For elements of $\mathbb{Z}$ we use the notation $a|b$ to mean that there is a $c \in \mathbb{Z}$ such that $b = ac$.

1. Let $a, b, c \in \mathbb{Z}$. Prove each of the following.
   (i) $a|a$.
   (ii) If $a|b$ and $b|a$, then $a = \pm b$.
   (iii) If $a|b$ and $b|c$, then $a|c$.
   (iv) If $ac|bc$ and $c \neq 0$, then $a|b$.
   (v) If $a|b$, then $ac|bc$.
   (vi) If $a|b$ and $a|c$, then $a|bx + cy$ for all $x, y \in \mathbb{Z}$.

2. The Fibonacci sequence is defined iteratively by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ $(n = 2, 3, \dots)$. Show that if m, $n \in \mathbb{N}$ satisfy $m|F_n$ and $m|F_{n+1}$, then $m = 1$.

3. Prove that if n is odd, then $8|n^2 - 1$.

4. Find all solutions x, $y \in \mathbb{Z}$ to the equation $x^2 - y^2 = 105$.

5. (i) Show that if m and n are integers of the form $4k + 1$, then so is mn.
   (ii) Show that if $m, n \in \mathbb{N}$, and mn is of the form $4k - 1$, then so is one of m and n.
   (iii) Show that every number of the form $4k - 1$ has a prime factor of this form.
   (iv) Show that there are infinitely many primes of the form $4k - 1$.