MATH 467 FACTORIZATON AND PRIMALITY TESTING, FALL TERM 2023, PRACTICE EXAM 1 SOLUTIONS

Note: Exam 1 will be 9:05-9:55, Wednesday 27th September 2023 Room 158 Willard

1. (25 marks) Prove that if $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then there are integers a, b such that $(a, b) = m$ and $ab = mn$ if and only if $m|n$.

If $(a, b) = m$ and $ab = mn$, then $m|a$ and $m|b$ so $m^2|mn$, whence $m|n$. On the other hand, if $m|n$, then take $a = m$ and $b = n$. Then $m|a$, $m|b$, so that $m|(a, b)$, and $(a, b)|a = m$. Thus $m = (a, b)$ and $ab = mn$.

2. (25 marks) Find all pairs of integers x and y such that $922x + 2163y = 7$.

$2163 - 2.922 = 319$, $922 - 2.319 = 284$, $319 - 284 = 35$, $284 - 8.35 = 4$, $35 - 8.4 = 3$, $4 - 1.3 = 1$. $1 = 4 - 1.3 = 4 - (35 - 8.4) = 9(284 - 8.35) - 35 = 9.284 - 73(319 - 284) = 82(922 - 2.319) - 73.319 = 82.922 - 237(2163 - 2.922) = 556.922 - 237.2163$. $7 = 3892.922 - 1659.2163 = 1729.922 - 737.2163$. The general solution is $x = 1729 + 2163t$, $y = -737 - 2163t$.

3. (25 marks) Prove that $1365|n^{13} - n$.

We make repeated use of Fermat's Little Theorem. $1365 = 3.5.7.13$. $n^{13} = (n^3)^4 n \equiv n^5 \equiv n^3 n^2 \equiv n^3 \equiv n \pmod 3$, $n^{13} = (n^5)^2 n^3 \equiv n^5 \equiv n \pmod 5$, $n^{13} = n^7 n^6 \equiv n^7 \equiv n \pmod 7$, $n^{13} \equiv n \pmod{13}$.

4. (25 marks) Prove that 23n-1 is never a perfect square.

The perfect squares x^2 modulo 23 satisfy $x^2 \equiv (23 - x)^2 \equiv x^2 \pmod{23}$, so the only non zero residue classes which contain perfect squares are given by x^2 with $1 \le x \le 11$ and are

$$1^2 \equiv 1, 2^2 \equiv 4, 3^2 \equiv 9, 4^2 \equiv 16, 5^2 \equiv 2, 6^2 \equiv 13,$$

$$7^2 = 49 \equiv 3, 8^2 \equiv 18, 9^2 \equiv 12, 10^2 \equiv 8, 11^2 \equiv 6.$$

$-1 \equiv 22 \pmod{23}$ is not in the list.