Introduction to Number Theory Chapter 3 Congruences and Residue Classes

Robert C. Vaughan

January 21, 2025

Residue Classes

- The next topic was first developed by Gauss.

Definition 1

Let $m \in \mathbb{N}$ and define *the residue class* $\overline{r}$ modulo m by

$$\overline{r} = \{x \in \mathbb{Z} : m|(x-r)\}.$$

By the division algorithm every integer is in one

$$\overline{0}, \overline{1}, \ldots, \overline{m-1}.$$

This is often called a *complete* system of residues modulo m.

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This is often called a *complete* system of residues modulo m.

- The remarkable thing is that we can perform arithmetic on the residue classes just as if they were numbers.
- The residue class $\overline{0}$ behaves like the number 0,
- because $\overline{0}$ is the set of multiples of m and adding any one of them to an element of $\overline{r}$ does not change the remainder.

- Thus for any r

$$\overline{0} + \overline{r} = \overline{r} = \overline{r} + \overline{0}.$$

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- Suppose that we are given any two residue classes $\overline{r}$ and $\overline{s}$ modulo m. Let t be the remainder of $r + s$ on division by m. Then the elements of $\overline{r}$ and $\overline{s}$ are of the form $r + mx$ and $s + my$ and we know that $r + s = t + mz$ for some z.

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- Thus $r + mx + s + my = t + m(z + x + y)$ is in $\overline{t}$, and it is readily seen that the converse is true.
- Thus it makes sense to write $\overline{r} + \overline{s} = \overline{t}$, and then we have $\overline{r} + \overline{s} = \overline{s} + \overline{r}$.

- Thus for any r
$$\overline{0}+\overline{r}=\overline{r}=\overline{r}+\overline{0}.$$
- Suppose that we are given any two residue classes $\overline{r}$ and $\overline{s}$ modulo m. Let t be the remainder of $r+s$ on division by m. Then the elements of $\overline{r}$ and $\overline{s}$ are of the form $r+mx$ and $s+my$ and we know that $r+s=t+mz$ for some z.
- Thus $r+mx+s+my=t+m(z+x+y)$ is in $\overline{t}$, and it is readily seen that the converse is true.
- Thus it makes sense to write $\overline{r}+\overline{s}=\overline{t}$, and then we have $\overline{r}+\overline{s}=\overline{s}+\overline{r}$.
- One can also check that
$$\overline{r}+\overline{-r}=\overline{0}.$$

- In connection with this Gauss introduced a notation.

Definition 2

Let $m \in \mathbb{N}$. If two integers x and y satisfy $m|x-y$, then we write

$$x \equiv y \pmod{m}$$

and we say that x is *congruent to* y modulo m.

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- Here are some of the properties of congruences.

$$x \equiv x \pmod{m},$$

$$x \equiv y \pmod{m} \text{ iff } y \equiv x \pmod{m},$$

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- These say that the relationship $\equiv$ is reflexive, symmetric and transitive.
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- It follows that congruences modulo m partition the integers into equivalence classes.

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- If f is a polynomial with integer coefficients, and $x \equiv y \pmod{m}$, then $f(x) \equiv f(y) \pmod{m}$.
- Wait a minute, this means that one can use congruences just like doing arithmetic on the integers!

- The following tells us something about this structure.

Theorem 3

Suppose that $m \in \mathbb{N}$, $k \in \mathbb{Z}$, $(k, m) = 1$ and

$$\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_m$$

forms a complete set of residues modulo m. Then so does

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- If they were the same integer, than $ka_i + mx = ka_j + my$, so that $k(a_i - a_j) = m(y - x)$.
- But then $m|k(a_i - a_j)$ and since $(k, m) = 1$ we would have $m|a_i - a_j$ so $\overline{a}_i$ and $\overline{a}_j$ would be identical residue classes, so $i = j$.

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- In connection with this we introduce Euler's function.

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A function defined on $\mathbb{N}$ is called an arithmetical function.

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Euler's function $\phi(n)$ is the number of $x \in \mathbb{N}$ with $1 \le x \le n$ and $(x, n) = 1$.

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A set of $\phi(m)$ distinct residue classes $\overline{r}$ modulo m with $(r, m) = 1$ is called a set of *reduced* residues modulo m.

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- Since $(1, 1) = 1$ we have $\phi(1) = 1$.
- If p is prime, then the x with $1 \leq x \leq p - 1$ satisfy $(x, p) = 1$, but $(p, p) = p \neq 1$. Hence $\phi(p) = p - 1$.

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- The numbers x with $1 \le x \le 30$ and $(x, 30) = 1$ are $1, 7, 11, 13, 17, 19, 23, 29$, so $\phi(30) = 8$.

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- What is left are the $\phi(m)$ *reduced* fractions with denominator m.
- Suppose instead of removing the non-reduced ones we just write them in their lowest form.
- Then for each divisor k of m we obtain all the reduced fractions with denominator k.

- In fact we just proved the following.

Theorem 7

For each $m \in \mathbb{N}$ we have

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- We just saw that $\phi(1) = 1$, $\phi(p) = p - 1$, $\phi(30) = 8$

Example 8

The divisors of 30 are $1, 2, 3, 5, 6, 10, 15, 30$ and

$$\phi(6) = 2,\ \phi(10) = 4, \phi(15) = 8$$

so

$$\sum_{k|30} \phi(k) = 1 + 1 + 2 + 4 + 2 + 4 + 8 + 8 = 30.$$

- Now we can prove a companion theorem to Theorem 3 for reduced residue classes.

Theorem 9

Suppose that $(k, m) = 1$ *and that*

$$a_1, a_2, \ldots, a_{\phi(m)}$$

forms a set of reduced residue classes modulo m*. Then*

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also forms a set of reduced residues modulo m.

- **Proof.** In view of the earlier theorem the residue classes ka_j are distinct, and since $(a_j, m) = 1$ we have $(ka_j, m) = 1$ so they give $\phi(m)$ distinct reduced residue classes, so they are all of them in some order.

- We now examine the structure of residue systems.

Theorem 10

Suppose m, $n \in \mathbb{N}$ and $(m,n)=1$, and consider the $xn+ym$ with $1 \le x \le m$ and $1 \le y \le n$. Then they form a complete set of residues modulo mn. If in addition x and y satisfy $(x,m)=1$ and $(y,n)=1$, then they form a reduced set.

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- **Proof.** If $xn + ym \equiv x'n + y'm \pmod{mn}$, then $xn \equiv x'n \pmod{m}$, so $x \equiv x' \pmod{m}$, $x = x'$. Likewise $y = y'$.

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- In the restricted case $(xn + ym, m) = (xn, m) = (x, m) = 1$ and likewise $(xn + ym, n) = 1$, so $(xn + ym, mn) = 1$ and the $xn + ym$ all belong to reduced residue classes.

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Suppose $m,\ n \in \mathbb{N}$ and $(m,n)=1$, and consider the $xn+ym$ with $1 \le x \le m$ and $1 \le y \le n$. Then they form a complete set of residues modulo mn. If in addition x and y satisfy $(x,m)=1$ and $(y,n)=1$, then they form a reduced set.

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- Now let $(z,mn)=1$. Choose x',y',x,y so that $x'n+y'm=1$, $x \equiv x'z \pmod{m}$ and $y \equiv y'z \pmod{n}$.

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- Now let $(z, mn) = 1$. Choose x', y', x, y so that $x'n + y'm = 1$, $x \equiv x'z \pmod{m}$ and $y \equiv y'z \pmod{n}$.
- Then $xn + ym \equiv x'zn + y'zm = z \pmod{mn}$ and hence every reduced residue is included.

- Here is a table of $xn + ym \pmod{mn}$ when $m = 5,\ n = 6$.

Example 11

x / y	1	2	3	4	5
1	11	17	23	29	5
2	16	22	28	4	10
3	21	27	3	9	15
4	26	2	8	14	20
5	1	7	13	19	25
6	6	12	18	24	30

The 30 numbers 1 through 30 appear exactly once each. The 8 reduced residue classes occur precisely in the intersection of rows 1 and 5 and columns 1 through 4.

- Immediate from Theorem 10 we have

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If $(m, n) = 1$, *then* $\phi(mn) = \phi(m)\phi(n)$.

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Definition 13

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whenever $(m, n) = 1$ we say that f is *multiplicative*.

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- Thus we have another

Corollary 14

Euler's function is multiplicative.

This enables a full evaluation of $\phi(n)$.

- If $n = p^k$, then the number of reduced residue classes modulo p^k is the number of x with $1 \le x \le p^k$ and $p \nmid x$.

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- Some special cases.

Example 16

We have $\phi(9) = 6$, $\phi(5) = 4$, $\phi(45) = 24$. Note that $\phi(3) = 2$ and $\phi(9) \ne \phi(3)^2$.

- Here is a beautiful and useful theorem.

Theorem 17 (Euler)

Suppose that $m \in \mathbb{N}$ and $a \in \mathbb{Z}$ with $(a, m) = 1$. Then

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

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Corollary 18 (Fermat)

Let p be a prime and $a \in \mathbb{Z}$. Then $a^p \equiv a \pmod{p}$. If $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.

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$$a^{560} \equiv 1 \pmod{561}$$

for *all* a with $(a, 561) = 1$.

- Such numbers are interesting

Definition 19

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- Thus for $M(n)$ to be prime it is necessary that n be prime.

Example 21

We have $3 = 2^2 - 1$, $7 = 2^3 - 1$, $31 = 2^5 - 1\,127 = 2^7 - 1$. However that is not sufficient. $2^{11} - 1 = 2047 = 23 \times 89$.

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Theorem 22

The congruence $ax \equiv b \pmod{m}$ is soluble iff $(a, m)|b$, and the general solution is given by a residue class x_0 modulo $m/(a, m)$. x_0 can be found by applying Euclid's algorithm.

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- Let x_0, y_0 be such a solution and let x, y be any solution. Then $a/(a, m)(x - x_0) \equiv 0 \pmod{m/(a, m)}$ and since $(a/(a, m), m/(a, m)) = 1$ it follows that x is in the residue class $x_0 \pmod{m/(a, m)}$.

- A curious result which uses somewhat similar ideas.

Theorem 23 (Wilson)

Let p be a prime number, then $(p-1)! \equiv -1 \pmod{p}$.

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- However this is useless since $(p-1)!$ grows very rapidly.

- What about simultaneous linear congruences?

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- Thus it is convenient to assume $(m_i, m_j) = 1$ when $i \neq j$.

- The following is known as the Chinese Remainder Theorem

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Suppose that $(m_i, m_j) = 1$ for every $i \neq j$. Then the system (2.2) has as its complete solution precisely the members of a unique residue class modulo $m_1 m_2 \dots m_r$.

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- Then for every j, since $m_j | M_i$ when $i \neq j$ we have

$$x \equiv N_j M_j \pmod{m_j}$$
$$\equiv c_j \pmod{m_j}$$

so the residue class $x \pmod{M}$ gives a solution.

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- Then for every j we have

$$\begin{aligned} y &\equiv c_j \pmod{m_j} \\ &\equiv x \pmod{m_j} \end{aligned}$$

and so $m_j | y - x$.

- $$\begin{cases} x & \equiv c_1 \pmod{m_1}, \\ \dots & \dots \\ x & \equiv c_r \pmod{m_r} \end{cases}$$
- Now we have to show that the solution modulo M is unique.
- Suppose y is also a solution of the system.
- Then for every j we have

$$\begin{aligned} y &\equiv c_j \pmod{m_j} \\ &\equiv x \pmod{m_j} \end{aligned}$$

  and so $m_j | y - x$.
- Since the m_j are pairwise co-prime we have $M | y - x$, so y is in the residue class x modulo M.

- Consider

Example 25

$$x \equiv 3 \pmod{4},$$
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- $m_1 = 4,\ m_2 = 21,\ m_3 = 25,\ M = 2100,\ M_1 = 525,$ $M_2 = 100,\ M_3 = 84.$ Thus first we have to solve

$$525N_1 \equiv 3 \pmod{4},$$
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- Reducing the constants gives
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- Thus we can take $N_1 = 3$, $N_2 = 20$, $7 \equiv -18 \pmod{25}$ so $N_3 \equiv -2 \equiv 23 \pmod{25}$. Then the complete solution is

$$\begin{aligned} x &\equiv N_1M_1 + N_2M_2 + N_3M_3 \\ &= 3 \times 525 + 20 \times 100 + 23 \times 84 \\ &= 5507 \\ &\equiv 1307 \pmod{2100}. \end{aligned}$$

- The solution of a general polynomial congruence can be quite tricky, even for a polynomial with a single variable

$$f(x) := a_0 + a_1 x + \cdots + a_j x^j + \cdots a_J x^J \equiv 0 \pmod{m} \quad (3.3)$$

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- That is, more than the degree 2. However, when the modulus is prime we have a more familiar conclusion.

- When we have a solution x to a polynomial congruence such as (3.3) we may sometimes refer to such values as a *root* of the polynomial modulo m.

Theorem 26 (Lagrange)

Suppose that p is prime, and $f(x) = a_0 + a_1x + \cdots + a_jx^j + \cdots$ is a polynomial with integer coefficients a_j and it has degree k modulo p. Then the number of incongruent solutions of

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- **Proof.** Degree 0 is obvious so we suppose $k \geq 1$.
- We use induction on the degree k.
- If a polynomial f has degree 1 modulo p, so that $f(x) = a_0 + a_1x$ with $p \nmid a_1$, then the congruence becomes $a_1x \equiv -a_0 \pmod{p}$ and since $a_1 \not\equiv 0 \pmod{p}$ (because f has degree 1) we know that this is soluble by precisely the members of a unique residue class modulo p.

- Now suppose that the conclusion holds for all polynomials of a given degree k and suppose that $f = a_0 + \cdots + a_{k+1}x^{k+1}$ has degree $k + 1$.

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- By the inductive hypothesis there are at most k possibilities for x_1, so at most $k+1$ in all.

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- The quadratic case we will need and will look at later.