Introduction to Number Theory Chapter 2 Euclid's Algorithm and Applications

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- Moreover this solution gives a very efficient algorithm and it is still the basis for many numerical methods in arithmetical applications.
- We may certainly suppose that a and $b > 0$ since multiplying either by (-1) does not change the (a, b) - we can replace x by $-x$ and y by $-y$.

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- Now apply the division algorithm iteratively as follows

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- That is, we stop the moment that there is a remainder equal to 0.
- This could be r_1 if $b|a$, for example, although the way it is written out above it is as if s is at least 3.
- The important point is that because $r_j < r_{j-1}$, sooner or later we must have a zero remainder.

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- On the other hand, starting at the bottom line $r_{s-1}|r_{s-2}$, $r_{s-1}|r_{s-3}$ and so on until we have $r_{s-1}|b$ and $r_{s-1}|a$. Recall that this means that $r_{s-1}|(a, b)$.

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- Thus we have just proved that

$$r_{s-1}|(a, b), \quad (a, b)|r_{s-1}, \quad r_{s-1} = (a, b).$$

- Consider.

Example 1

Let $a = 10678,\ b = 42$

$$10678 = 42 \times 254 + 10$$
$$42 = 10 \times 4 + 2$$
$$10 = 2 \times 5.$$

Thus $(10678, 42) = 2$.

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- In general this is tedious and computationally wasteful since it requires all our calculations to be stored.

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- Define $x_{-1} = 1$, $y_{-1} = 0$, $x_0 = 0$, $y_0 = 1$ and then lay the calculations out as follows.

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- Suppose $r_j = ax_j + by_j$ is established for all $j \le k$. Then

$$\begin{aligned}
r_{k+1} &= r_{k-1} - q_{k+1} r_k\\
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- In particular $(a, b) = r_{s-1} = ax_{s-1} + by_{s-1}$.

- Hence laying out the example above in this expanded form we have

$$r_{-1} = 10678,\ r_0 = 42,\ x_{-1} = 1,\ x_0 = 0,\ y_{-1} = 0,\ y_0 = 1,$$

$$\begin{array}{lll} 10678 = 42 \cdot 254 + 10, & x_1 = 1, & y_1 = -254 \\ 42 = 10 \cdot 4 + 2, & x_2 = -4, & y_2 = 1017 \\ 10 = 2 \cdot 5. & & \end{array}$$

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- It is also possible to set this up using matrices.

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- If the s-th row is the last one to be computed, calculate $q_s = \lfloor r_{s-1}/r_s \rfloor$.
- Then take the last two rows computed and pre multiply by $(1, -q_s)$

$$(1, -q_s) \begin{pmatrix} r_{s-1}, & x_{s-1}, & y_{s-1} \\ r_s, & x_s, & y_s \end{pmatrix} = (r_{s+1}, x_{s+1}, y_{s+1})$$

to obtain the $s+1$-st row.

- Here is a simple example.

Example 2

Let $a = 4343$, $b = 973$. We can lay this out as follows

	4343	1	0
4	973	0	1
2	451	1	−4
6	71	−2	9
2	25	13	−58
1	21	−28	125
5	4	41	−183
	1	−233	1040

Thus $(4343, 973) = 1 = (-233)4343 + (1040)973$.

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- Thus it makes sense to suppose that one of a or b is non-zero.
- Then since (a, b) divides the left hand side, we can only have solutions if $(a, b)|c$.

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- Then as $\left(\frac{a}{(a,b)}, \frac{b}{(a,b)}\right) = 1$ we have by an earlier example that $y_0 - y = z\frac{a}{(a,b)}$ and $x - x_0 = z\frac{b}{(a,b)}$ for some z.

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- But any x and y of this form give a solution, so we have found the complete solution set.

- We have

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Suppose that a and b are not both 0 and $(a, b)|c$. Suppose further that $ax_0 + by_0 = c$. Then every solution of

$$ax + by = c$$

is given by

$$x = x_0 + z\frac{b}{(a, b)}, \quad y = y_0 - z\frac{a}{(a, b)}$$

where z is any integer.

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$$ax + by = c$$

is given by

$$x = x_0 + z\frac{b}{(a, b)}, \quad y = y_0 - z\frac{a}{(a, b)}$$

where z is any integer.

- One can see here that the solutions x all leave the same remainder on division by $\frac{b}{(a,b)}$ and likewise for y on division by $\frac{a}{(a,b)}$. This suggests that there may be a useful way of classifying integers.