Number Theory Chapter 0

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Introduction to Number Theory

- Number theory in its most basic form is the study of the set of *integers*
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$$
and its important subset
$$\mathbb{N} = \{1, 2, 3, \ldots\},$$
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- There are many fundamental but unresolved questions. A simple example concerns the sequence of twin primes,
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- There are many fundamental but unresolved questions. A simple example concerns the sequence of twin primes,
$$3, 5; 5, 7; 11, 13; 17, 19; 29, 31; 41, 43; 59, 61; 71, 73; \ldots.$$
- There seem to be many such pairs of primes. Are there infinitely many? Perhaps, but we do not know how to prove it.

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- This is a *proofs* based course. The proofs will be mostly short and simple, but they are necessary, and as a general principle understanding the proof usually reveals the underlying structure which is the reason why the theorem is true.
- There is an instructive example due to J. E. Littlewood in 1912.

Littlewood

- Let $\pi(x)$ denote the number of prime numbers not exceeding x. Gauss had suggested that

$$\int_0^x \frac{dt}{\log t}$$

should be a good approximation to $\pi(x)$

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- Here is a table of values which illustrates this for various values of x out to 10^{22}.

x	$\pi(x)$	$\mathrm{li}(x)$
10^4	1229	1245
10^5	9592	9628
10^6	78498	78626
10^7	664579	664917
10^8	5761455	5762208
10^9	50847534	50849233
10^{10}	455052511	455055613
10^{11}	4118054813	4118066399
10^{12}	37607912018	37607950279
10^{13}	346065536839	346065645809
10^{14}	3204941750802	3204942065690
10^{15}	29844570422669	29844571475286
10^{16}	279238341033925	279238344248555
10^{17}	2623557157654233	2623557165610820
10^{18}	24739954287740860	24739954309690413
10^{19}	234057667276344607	234057667376222382
10^{20}	2220819602560918840	2220819602783663483
10^{21}	21127269486018731928	21127269486616126182
10^{22}	201467286689315906290	201467286691248261498

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- We now believe that the first sign change occurs when

$$x \approx 1.387162 \times 10^{316} \tag{1.1}$$

  well beyond what can be calculated directly.

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- Let me turn back to that table, as it illustrates something else very interesting.

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The Riemann Hypothesis

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- There is a million dollar prize for a proof, or a disproof. And probably an automatic professorship at the most prestigious universities for anyone who wins it.
- By the way, one might wonder if there is something random in the distribution of the primes. This is how random phenomena are supposed to behave.