Math 465 Number Theory, Spring 2025, Solutions 14

1. For $x \ge 0$ define $B(x) = \lfloor x \rfloor - \lfloor x/2 \rfloor - \lfloor x/3 \rfloor - \lfloor x/5 \rfloor + \lfloor x/30 \rfloor$.
(i) Prove that $B(x)$ is periodic with period 30,

$$B(x) = \begin{cases} 0 & x \in [0,1), \\ 1 & x \in [1,6), \\ 0 & x \in [6,7), \\ 1 & x \in [7,10), \end{cases} \qquad B(x) = \begin{cases} 0 & x \in [10,11), \\ 1 & x \in [11,12), \\ 0 & x \in [12,13), \\ 1 & x \in [13,15). \end{cases}$$

and that if $0 \le x < 15$, then $B(x+15) = B(x) + \lfloor x/2 \rfloor - \lfloor (x+1)/2 \rfloor$. Deduce that $0 \le B(x) \le 1$ for all x.

$B(x+30) = B(x) + 30 - 15 - 10 - 6 + 1 = B(x)$. When $0 \le x < 15$ one can evaluate $B(x)$ by hand. Also, we have $B(x+15) = \lfloor x \rfloor + 15 - \lfloor (x+1)/2 \rfloor - 7 - \lfloor x/3 \rfloor - 5 - \lfloor x/5 \rfloor - 3 + 0 = B(x) + \lfloor x/2 \rfloor - \lfloor (x+1)/2 \rfloor$. Let $C(x) = \lfloor x/2 \rfloor - \lfloor (x+1)/2 \rfloor$. Then $C(x)$ is periodic with period 2, and $C(x) = 0$ when $2k-2 \le x < 2k-1$ and is -1 when $2k-1 \le x < 2k$. Thus when $0 \le x < 15$ we have $0 \le B(x+15) \le 1$. Hence $0 \le B(x) \le 1$ for all x with $0 \le x < 30$, and so, by periodicity, for all x.

(ii) Let $T(x) = \displaystyle\sum_{m \le x} \Lambda(m) B(x/m)$. Prove that $T(x) = c'x + O(\log x)$ where $c' = \dfrac{1}{2}\log 2 + \dfrac{1}{3}\log 3 + \dfrac{1}{5}\log 5 - \dfrac{1}{30}\log 30 = 0.9212\ldots$.

By Theorem 6.16 we have

$$\sum_{m \le x} \Lambda(m) \left\lfloor \frac{x/k}{m} \right\rfloor = \sum_{m \le x/k} \Lambda(m) \left\lfloor \frac{x/k}{m} \right\rfloor = \frac{x}{k}\left(\log\frac{x}{k} - 1\right) + O\left(\log\frac{x}{k}\right).$$

Thus $T(x) = x(\log x - 1) - \frac{x}{2}\left(\log\frac{x}{2} - 1\right) - \frac{x}{3}\left(\log\frac{x}{3} - 1\right) - \frac{x}{5}\left(\log\frac{x}{5} - 1\right) + \frac{x}{30}\left(\log\frac{x}{3}0 - 1\right) + O(\log x)$ and on simplification this is $c'x + O(\log x)$.

(iii) Prove that $\psi(x) - \psi(x/6) \le T(x) \le \psi(x)$.

By (i) we have $B(x) \le 1$ when $x \ge 1$. Thus $T(x) \le \psi(x)$. Also for all m we have $B(x/m) \ge 0$, and for all m with $x/6 < m \le x$ we have $B(x/6) = 1$. Hence $T(x) \ge \sum_{x/6 < m \le x} \Lambda(m) = \psi(x) - \psi(x/6)$.

(iv) Prove that if $x \ge 2$, then

$$c'x + O(\log x) \le \psi(x) \le \frac{6c'}{5}x + O(\log^2 x).$$

The lower bound is immediate from the upper bound in (iii) and (ii). For the upper bound observe that by the lower bound in (iii) $\psi(x/6^{k-1}) - \psi(x/6^k) \le T(x/6^{k-1})$ and then sum over k.