MATH 465 NUMBER THEORY, SPRING 2025, SOLUTIONS 13

1. A number $n \in \mathbb{N}$ is *squarefree* when it has no repeated prime factors. For $X \in \mathbb{R}$, $X \geq 1$ let $Q(X)$ denote the number of squarefree numbers not exceeding X. (i) Prove that $\displaystyle\sum_{\substack{m \\ m^2|n}} \mu(m) = \begin{cases} 1 & \text{when } n \text{ is squarefree} \\ 0 & \text{otherwise.} \end{cases}$

(ii) Prove that $\displaystyle Q(x) = \sum_{m \leq \sqrt{x}} \mu(m) \left\lfloor \frac{x}{m^2} \right\rfloor$.

(iii) Prove that $Q(X) = \dfrac{6}{\pi^2} X + O\left(\sqrt{X}\right)$. (You can assume that $\sum_{m=1}^{\infty} \mu(m) m^{-2} = 6/\pi^2$.)

(i)Let n_1^2 be the largest square dividing n. Then $m^2|n$ iff $m|n_1$. (ii) $Q(x) = \sum_{n \leq x} \sum_{m^2|n} \mu(m)$ and interchange the order and replace n by km^2. (iii) In (ii) note that $\lfloor x/m^2 \rfloor$ differs from x/m^2 by at most 1 and that $|\sum_{n > \sqrt{x}} \mu(m) m^{-2}| \leq x^{-1/2} + \int_{\sqrt{x}}^{\infty} t^{-2} dt$.

2. Assume the same notation as in question 1. (i) Prove that if $n \in \mathbb{N}$, then $\displaystyle Q(n) \geq n - \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor$. (ii) Prove that $\sum_p \frac{1}{p^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{4k(k+1)} = \frac{1}{2}$. (iii) Prove that $Q(n) > n/2$ for all $n \in \mathbb{N}$. (iv) Prove that every $n > 1$ is a sum of two squarefree numbers.

(i) If n is not squarefree, then it is divisible by the square of some prime. Thus the number of non-squarefree numbers not exceeding n is $\leq \sum_p \lfloor n/p^2 \rfloor$. (ii) Every prime is either 2 or of the form $2k+1$ (with k omitting some values such as 4), so we have strict inequality. (iii) At once from (i) and (ii). (iv) Pigeon hole principle. Associate a box with each integer in $[1, n-1]$. Put each squarefree number m in its box, and put each $n - m$ with m squarefree and $1 \leq m \leq n-1$ in its box. The number of objects in boxes is $2Q(n-1)$. By (iii) $Q(n-1) > n-1$ and so at least one box contains two objects m and $n - m'$. Hence $m = n - m'$.

3. Let $f(n)$ denote the number of solutions of $x^3 + y^3 = n$ in natural numbers x, y. Show that $\sum_{n\le X} f(n) = AX^{2/3} + O\left(X^{1/3}\right)$ where $A = \int_0^1 (1-\alpha^3)^{1/3} d\alpha$.

We are counting the number of integral lattice points x, y with $X > 0$, $y > 0$ and $x^3 + y^3 \le X$. This is the area of that region with an error the length of the boundary, which is of length $\ll X^{1/3}$. The area is

$$\int_0^{X^{1/3}} (X - t^3)^{1/3} dt = X^{2/3} A$$

by the change of variable $t = X^{1/3}\alpha$. Alternatively write the sum as

$$\sum_{x \le X^{1/3}} \lfloor (X - x^3)^{1/3} \rfloor = \sum_{x \le X^{1/3}} (X - x^3)^{1/3} + O(X^{1/3})$$

and replace the sum by an integral using monotonicity. A third possibility is to notice the difference in the number of lattice points between when we count out to $X^{1/3} - \sqrt{2}$ and $X^{1/3} + \sqrt{2}$.

4. Let $n \in \mathbb{N}$ and p be a prime number, show that the largest t such that $p^t | n!$ satisfies $t = \sum_{h=1}^{\infty} \left\lfloor \frac{n}{p^h} \right\rfloor$.

The exact power t_m to which p divides m is

$$t_m = \sum_{\substack{h=1 \\ p^h | m}}^{\infty} 1.$$

Hence

$$t = \sum_{m=1}^{n} t_m = \sum_{m=1}^{n} \sum_{\substack{h=1 \\ p^h | m}}^{\infty} 1 = \sum_{h=1}^{\infty} \sum_{\substack{m=1 \\ p^h | m}}^{n} 1 = \sum_{h=1}^{\infty} \left\lfloor \frac{n}{p^h} \right\rfloor .$$

An alternative proof is to observe that

$$\begin{aligned} n! &= \exp\left(\sum_{m=1}^{n} \log m \right) \\ &= \exp\left(\sum_{k=1}^{\infty} \Lambda(k) \left\lfloor \frac{n}{k} \right\rfloor \right) \\ &= \exp\left(\sum_{p,h} \log p \left\lfloor \frac{n}{p^h} \right\rfloor \right) \\ &= \prod_{p} p^{\sum_{h=1}^{\infty} \left\lfloor \frac{n}{p^h} \right\rfloor}. \end{aligned}$$