Math 465 Number Theory, Spring 2025, Solutions 12

A totally multiplicative arithmetical function $\chi : \mathbb{N} \to \mathbb{C}$ which is periodic modulo q, where $q \in \mathbb{N}$, and which satisfies $\chi(n) = 0$ when $(n, q) > 1$, is called a Dirichlet character.

1. Prove that if $(n, q) = 1$, then $\chi(n)^{\phi(q)} = 1$.

Since χ is multiplicative we have $\chi(1) = 1$. Hence, by Euler, $1 = \chi(n^{\phi(q)}) = \chi(n)^{\chi(q)}$.

2. (i) Prove that if χ is a Dirichlet character modulo q, then so is $\overline{\chi}$.
(ii) The Dirichlet character $\chi_0(n)$ modulo q which is 1 whenever $(n, q) = 1$ is called the principal character modulo q. Prove that if χ is a Dirichlet character modulo q, then $\chi\overline{\chi} = \chi_0$.

(i) We have $\overline{\chi}(mn) = \overline{\chi(mn)} = \overline{\chi(m)\chi(n)} = \overline{\chi}(m)\overline{\chi}(n)$ and $\overline{\chi}(n + mq) = \overline{\chi(n + mq)} = \overline{\chi(n)} = \overline{\chi}(n)$. (ii) If $(n, q) = 1$, then by question 1 $\chi(n)$ is a $\phi(q)$-th root of unity, so $\chi\overline{\chi}(n) = |\chi(n)| = 1$.

3. Prove that there are at most $\phi(q)^{\phi(q)}$ Dirichlet characters χ modulo q.

There are at most $\phi(q)$, $\phi(q)$-th roots of unity, so there are at most $\phi(q)$ possible values for $\chi(n)$ for each of the $\chi(q)$ reduced residue classes.

4. (i) Prove that if χ is a Dirichlet character modulo q and $(m, q) = 1$, then the numbers $\chi(mn)$ $(1 \le n \le q)$ are just a rearrangement of the numbers $\chi(n)$ $(1 \le n \le q)$.
(ii) Prove that if $(m, q) = 1$, then

$$\sum_{n \bmod q} \chi(n) = \chi(m) \sum_{n \bmod q} \chi(n)$$

(iii) Prove that if there is an m with $(m, q) = 1$ such that $\chi(m) \ne 1$, then

$$\sum_{n \bmod q} \chi(n) = 0,$$

and otherwise the sum is $\phi(q)$, i.e. when $\chi = \chi_0$.

(i) This follows at once from the observation that mn runs over a reduced set of residue as n does. (ii) By (i) the sum on the left equals

$$\sum_{n \bmod q} \chi(mn).$$

(iii) At once by (ii), if $\chi(m) \ne 1$ for some m, then the sum on either side is 0. If $\chi(m) = 1$ for every m with $(m, q) = 1$, then the sum is $\phi(q)$.

5. (i) Prove that if χ_1 and χ_2 are Dirichlet characters modulo q_1 and q_2 respectively, then $\chi(n) = \chi_1(n)\chi_2(n)$ is a Dirichlet character modulo q_1q_2.

(ii) Prove that if χ, χ_1, χ_2 are Dirichlet characters modulo q, then $\chi\chi_1 = \chi\chi_2$ if and only if $\chi_1 = \chi_2$.
(iii) Let $D(q)$ denote the number of Dirichlet characters modulo q. Prove that if χ is a given character modulo q and χ_1 ranges over the $D(q)$ characters modulo q, then so does $\chi\chi_1$.
(iv) Prove that if $(n,q) = 1$ and χ is a character modulo q, then

$$\sum_{\chi_1} \chi_1(n) = \chi(n) \sum_{\chi_1} \chi_1(n)$$

where each sum is over the $D(q)$ characters modulo q.
(v) Prove that if $(n,q) = 1$ and there is character χ modulo q with $\chi(n) \neq 1$, then

$$\sum_{\chi_1} \chi_1(n) = 0$$

and if there is no such character χ, then the sum is $D(q)$.

(i) We have $\chi(mn) = \chi_1(mn)\chi_2(mn) = \chi_1(m)\chi_2(m)\chi_1(n)\chi_2(n) = \chi(m)\chi(n)$, $\chi(1) = \chi_1(1)\chi_2(1) = 1$, $\chi(n+kq_1q_2) = \chi_1(n+(kq_2)q_1)\chi_2(n+(kq_1)q_2) = \chi_1(n)\chi_2(n)$ $= \chi(n)$, and if $(n, q_1q_2) > 1$, then there is a p dividing n which also divides q_1 or q_2, so $\chi_1(n) = 0$ or $\chi_2(n) = 0$ and thus $\chi(n) = 0$. (ii) If $\chi\chi_1 = \chi\chi_2$, then multiply both sides by $\overline{\chi}$. In the opposite case multiply both sides by χ. (iii) If $\chi\chi_1 = \chi\chi_1'$, then by (ii) $\chi_1 = \chi_1'$. Thus the functions $\chi\chi_1$ are distinct. But there are $D(q)$ of them and they are all characters modulo q. (iv) By (iii) the sum on the left is

$$\sum_{\chi_1} \chi(n)\chi_1(n).$$

(v) The same argument as in 3(iii).