Math 465 Number Theory, Spring 2025, Solutions 11

1. We define $\sigma(n)$ for $n \in \mathbb{N}$ to be the sum of the divisors of n, $\sigma(n) = \sum_{m|n} m$. (i) Prove that σ is a multiplicative function. (ii) Evaluate $\sigma(1050)$. (iii) Prove that $\sum_{m|n} \phi(m)\sigma(n/m) = nd(n)$. (iv) Show that if $\sigma(n)$ is odd, then n is a square or twice a square. (v) Prove that $\sum_{m|n} \mu(m)\sigma(n/m) = n$. (vi) Prove that $\sum_{m|n} \mu(n/m) \sum_{l|m} \mu(l)\sigma(m/l) = \phi(n)$.

(i) We have $\sigma = N * \mathbf{1}$ and N and $\mathbf{1}$ are multiplicative. (ii) $1050 = 2 \times 3 \times 5^2 \times 7$ and $\sigma(2) = 3$, $\sigma(3) = 4$, $\sigma(5^2) = 1 + 5 + 5^2 = 31$, $\sigma(7) = 8$, so $\sigma(1050) = 2976$. (iii) The LHS is $\phi * \sigma = (N * \mu) * (\mathbf{1} * N) = N * (\mu * \mathbf{1}) * N = N * e * N = N * N$ and $(N * N)(n) = \sum_{m|n} m \times (n/m) = nd(n)$. (iv) Suppose $n = p_1^{k_1} \dots p_s^{k_s}$. Since σ is multiplicative, each $\sigma(p_j^{k_j})$ will be odd. If p_j is odd, then $\sigma(p_j^{k_j} \equiv k_j + 1 \pmod 2$ and so k_j will be even. If, say, $p_1 = 2$, then we have a perfect square when k_1 is even and twice a perfect square when it is odd. (v) The LHS is $\mu * (\mathbf{1} * N) = (\mu * 1) * N = e * N = N$. (vi) The LHS is $\mu * \big(\mu * (\mathbf{1} * N)\big) = \mu * \big((\mu * \mathbf{1}) * N\big) = \mu * \big(e * N\big) = \mu * N = \phi$.

2. Suppose that $f(x)$ and $F(x)$ are complex-valued functions defined on $[1, \infty)$. Prove that $F(x) = \sum_{n \le x} f(x/n)$ for all x if and only if $f(x) = \sum_{n \le x} \mu(n)F(x/n)$ for all x.

If $F(x) = \sum_{n \le x} f(x/n)$, then

$$\sum_{n \le x} \mu(n)F(x/n) = \sum_{n \le x} \mu(n) \sum_{m \le x/n} f\big((x/n)/m\big) = \sum_{\ell \le x} \sum_{n|\ell} \mu(n)f(x/\ell) = f(x)$$

and if $f(x) = \sum_{n \le x} \mu(n)F(x/n)$, then

$$\sum_{n \le x} f(x/n) = \sum_{n \le x} \sum_{m \le x/n} \mu(m)F\big((x/n)/m\big) = \sum_{\ell \le x} F(x/\ell) \sum_{m|\ell} \mu(m) = F(x).$$

3. Show for each positive integer k that there is a unique arithmetic function ϕ_k such that $\sum_{m|n} \phi_k(m) = n^k$. Obtain a formula for $\phi_k(n)$ and show that $\phi_k(n)$ is multiplicative.

Let $\phi_k(n) = \sum_{m|n} m^k \mu(n/m)$ (1). Then by Möbius inversion ϕ_k satisfies the equation. Moreover if ϕ_k satisfies the equation, then by Möbius inversion it satisfies (1) and so is uniquely defined.

4. Suppose that the arithmetical function $\eta(n)$ satisfies $\sum_{m|n} \eta(m) = \phi(n)$. Show that $\eta(n)$ is multiplicative and evaluate $\eta(p^k)$.

By Möbius inversion $\eta = \phi * \mu$ and both ϕ and μ are multiplicative. Moreover $\eta(p^k) = \phi(p^k) - \phi(p^{k-1}) = p - 2$ when $k = 1$ and $p^{k-2}(p-1)^2$ when $k \ge 2$.

5. Evaluate $h(n) = \sum_{m|n} (-1)^m \mu(n/m)$.

We have $h(n) = \sum_{2|m|n} \mu(n/m) - \sum_{2\nmid m|n} \mu(n/m) =$
$2\sum_{2|m|n} \mu(n/m) - \sum_{m|n} \mu(n/m)$. If n is odd, then this is -1 when $n = 1$ and 0 when $n > 1$. If $2|n$, then this is $2\sum_{\ell|n/2} \mu\big((n/2)/\ell\big) - \sum_{m|n} \mu(n/m) = 0$ if $n > 2$ and $= 2$ when $n = 2$. Thus $h(1) = -1$, $h(2) = 2$ and $h(n) = 0$ when $n > 2$.