MATH 465 NUMBER THEORY,
SPRING TERM 2025, SOLUTIONS 4

1. Prove that when a natural number is written in the usual decimal notation, (i) it is divisible by 3 if and only if the sum if its digits is divisible by 3 and (ii) it is divisible by 9 if and only if the sum if its digits is divisible by 9.

We write $n = a_0 + 10a_1 + 10^2a_2 + \cdots + 10^k a_k$ and the usual decimal notation for n is then $a_k a_{k-1} \dots a_1 a_0$. Let $d = 3$ or 9. Then $10 \equiv 1 \pmod{d}$, so $10^j \equiv 1 \pmod{d}$. Hence $n \equiv a_0 + a_1 + \cdots + a_k \pmod{d}$ and so $n \equiv 0 \pmod{d}$ if and only if $a_0 + a_1 + \cdots + a_k \equiv 0 \pmod{d}$.

2. Solve $11x \equiv 21 \pmod{91}$.

$x \equiv 35 \pmod{91}$. There are many ways of doing this question. The systematic, but rather long, way to see this is to first use Euclid's algorithm to solve $11u+91v = (11, 91) = 1$. This gives, for example, $u = -33$ and $v = 4$. Multiplying by 21 gives $11x' + 91y' = 21$ (1) with $x' = -33.21 = -693$ and so $x \equiv -693 \pmod{91}$ gives a solution, and $-693 \equiv 35 \pmod{91}$. A shorter method is to observe that $(21, 91) = 7$ and so for (1) to hold we have $7|x' = 7t$ say. Then (1) becomes $11t + 13y' = 3$ and this can be seen to have the solution $t = 5$ and $y' = -4$. Thus (1) holds with $x' = 35$ and $y' = -4$. Hence the congruence is soluble with $x = 35$ and so $x \equiv 35 \pmod{91}$ is the general solution.

3. (i) Prove that x^3 lies in one of the residue classes $\overline{0}$, $\overline{1}$, $\overline{8}$ modulo 9.

(ii) Prove that if n is in one of the residue classes $\overline{4}$ or $\overline{5}$ modulo 9, then

$$x^3 + y^3 + z^3 = n$$

has no solution in integers x, y, z.

(i) Consider the 9 possible residue classes for x modulo 9; $0^3 \equiv 0$, $1^3 \equiv 1$, $2^3 \equiv 8$, $3^3 \equiv 0$, $4^3 \equiv 64 \equiv 1$, $5^3 \equiv (-4)^3 \equiv -1 \equiv 8$, $6^3 \equiv 0$, $7^3 \equiv -2^3 \equiv 1 \equiv 8$, $8^3 \equiv -1 \equiv 8$.

(ii) By (i) $x^3 + y^3 + z^3 \equiv$ modulo 9 to one of the combinations $0 + 0 + 0 \equiv 0$, $0+0+1 \equiv 1$, $0+0+8 \equiv 8$, $0+1+1 \equiv 2$, $0+1+8 \equiv 0$, $0+8+8 \equiv 7$, $1+1+1 \equiv 3$, $1 + 1 + 8 \equiv 1$, $1 + 8 + 8 \equiv$, $8 + 8 + 8 \equiv 6$. None of these combinations are $\equiv 4$ or 5.

4. Prove that if m, $n \in \mathbb{N}$ and $(m, n) = 1$, then $m^{\phi(n)} + n^{\phi(m)} \equiv 1 \pmod{mn}$.

We have $m^{\phi(n)} + n^{\phi(m)} \equiv n^{\phi(m)} \equiv 1 \pmod{m}$ by the Fermat–Euler theorem. Thus $m^{\phi(n)} + n^{\phi(m)} - 1 \equiv n^{\phi(m)} - 1 \equiv 0 \pmod{m}$ so m divides the LHS. Likewise modulo n. Since $(m, n) = 1$ so does mn.