465 Introduction to Number Theory, Spring Term 2025, Problems 14

Return by Monday 28th April

1. (Chebyshev). Define $B(x) = \lfloor x \rfloor - \lfloor x/2 \rfloor - \lfloor x/3 \rfloor - \lfloor x/5 \rfloor + \lfloor x/30 \rfloor$.

(i) Prove that $B(x)$ is periodic with period 30,

$$B(x) = \begin{cases} 0 & x \in [0,1), \\ 1 & x \in [1,6), \\ 0 & x \in [6,7), \\ 1 & x \in [7,10), \\ 0 & x \in [10,11), \\ 1 & x \in [11,12), \\ 0 & x \in [12,13), \\ 1 & x \in [13,15) \end{cases}$$

and that if $0 \le x < 15$, then $B(x+15) = B(x) + \lfloor x/2 \rfloor - \lfloor (x+1)/2 \rfloor$. Deduce that $0 \le B(x) \le 1$ for all x.

(ii) Let $T(x) = \sum_{m \le x} \Lambda(m) B(x/m)$. Prove that $T(x) = c'x + O(\log x)$ where $c' = \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{5}\log 5 - \frac{1}{30}\log 30 = 0.9212\ldots$.

(iii) Prove that $\psi(x) - \psi(x/6) \le T(x) \le \psi(x)$.

(iv) Prove that if $x \ge 2$, then

$$c'x + O(\log x) \le \psi(x) \le \frac{6c'}{5}x + O(\log^2 x).$$

Remark: $6c'/5 = 1.1054\ldots$.