MATH 465 NUMBER THEORY, SPRING 2025, PROBLEMS 13

Return by Monday 21st April

1. A number $n \in \mathbb{N}$ is *squarefree* when it has no repeated prime factors. For $X \in \mathbb{R}$, $X \geq 1$ let $Q(X)$ denote the number of squarefree numbers not exceeding X.

(i) Prove that $\displaystyle\sum_{\substack{m \\ m^2|n}} \mu(m) = \begin{cases} 1 & \text{when } n \text{ is squarefree,} \\ 0 & \text{otherwise.} \end{cases}$

(ii) Prove that $\displaystyle Q(x) = \sum_{m \leq \sqrt{x}} \mu(m) \left\lfloor \frac{x}{m^2} \right\rfloor$.

(iii) Prove that $\displaystyle Q(X) = \frac{6}{\pi^2} X + O\left(\sqrt{X}\right)$.

(You can assume that $\sum_{m=1}^{\infty} \mu(m) m^{-2} = 6/\pi^2$.)

2. Assume the same notation as in question 1.

(i) Prove that if $n \in \mathbb{N}$, then $\displaystyle Q(n) \geq n - \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor$.

(ii) Prove that

$$\sum_p \frac{1}{p^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{4k(k+1)} = \frac{1}{2}.$$

(iii) Prove that $Q(n) > n/2$ for all $n \in \mathbb{N}$.

(iv) Prove that every $n > 1$ is a sum of two squarefree numbers.

3. Let $f(n)$ denote the number of solutions of $x^3 + y^3 = n$ in natural numbers x, y. Show that

$$\sum_{n \leq X} f(n) = AX^{2/3} + O\left(X^{1/3}\right) \quad \text{where} \quad A = \int_0^1 (1 - \alpha^3)^{1/3} d\alpha.$$

(As an aside, observe that $A = \frac{1}{3} B(4/3, 1/3) = \frac{\Gamma(4/3)^2}{\Gamma(5/3)} = \frac{1}{\pi} 3^{3/2} \Gamma(4/3)^3$. Here $B(\alpha, \beta)$ is the Beta function.)

4. Let $n \in \mathbb{N}$ and p be a prime number, show that the largest t such that $p^t | n!$ satisfies

$$t = \sum_{h=1}^{\infty} \left\lfloor \frac{n}{p^h} \right\rfloor.$$