Math 465 Number Theory, Spring 2025, Problems 12

Return by Monday 14th April

A totally multiplicative arithmetical function $\chi : \mathbb{N} \to \mathbb{C}$ which is periodic modulo q, where $q \in \mathbb{N}$, and which satisfies $\chi(n) = 0$ when $(n,q) > 1$, is called a Dirichlet character. The Legendre and Jacobi symbols are examples.

1. Prove that if $(n,q) = 1$, then $\chi(n)^{\phi(q)} = 1$. [An example with $q = 5$ is given by $\chi(1) = 1$, $\chi(2) = i$, $\chi(3) = -i$, $\chi(4) = -1$, $\chi(5) = 0$.]

2. (i) Prove that if χ is a Dirichlet character modulo q, then so is $\overline{\chi}$.
(ii) The Dirichlet character $\chi_0(n)$ modulo q which is 1 whenever $(n,q) = 1$ is called the principal character modulo q. Prove that if χ is a Dirichlet character, then $\chi\overline{\chi} = \chi_0$.

3. Prove that there are at most $\phi(q)^{\phi(q)}$ Dirichlet characters χ modulo q.

4. (i) Prove that if χ is a Dirichlet character modulo q and $(m,q) = 1$, then the numbers $\chi(mn)$ $(1 \le n \le q)$ are just a rearrangement of the numbers $\chi(n)$ $(1 \le n \le q)$.
(ii) Prove that if $(m,q) = 1$, then

$$\sum_{n \bmod q} \chi(n) = \chi(m) \sum_{n \bmod q} \chi(n)$$

(iii) Prove that if there is an m with $(m,q) = 1$ such that $\chi(m) \ne 1$, then

$$\sum_{n \bmod q} \chi(n) = 0,$$

and otherwise the sum is $\phi(q)$, i.e. when $\chi = \chi_0$.

5. (i) Prove that if χ_1 and χ_2 are Dirichlet characters modulo q_1 and q_2 respectively, then $\chi(n) = \chi_1(n)\chi_2(n)$ is a Dirichlet character modulo $q_1 q_2$.
(ii) Prove that if χ, χ_1, χ_2 are Dirichlet characters modulo q, then $\chi\chi_1 = \chi\chi_2$ if and only if $\chi_1 = \chi_2$.
(iii) Let $D(q)$ denote the number of Dirichlet characters modulo q. Prove that if χ is a given character modulo q and χ_1 ranges over the $D(q)$ characters modulo q, then so does $\chi\chi_1$.
(iv) Prove that if $(n,q) = 1$ and χ is a character modulo q, then

$$\sum_{\chi_1} \chi_1(n) = \chi(n) \sum_{\chi_1} \chi_1(n)$$

where each sum is over the $D(q)$ characters modulo q.
(v) Prove that if $(n,q) = 1$ and there is character χ modulo q with $\chi(n) \ne 1$, then

$$\sum_{\chi_1} \chi_1(n) = 0$$

and if there is no such character χ, then the sum is $D(q)$. [Note the parallel between the residues and the characters. It turns out that $D(q) = \phi(q)$ and the Dirichlet characters form a group under multiplication which is isomorphic to the multiplicative group of reduced residue classes modulo q.]