Math 465 Number Theory, Spring 2025, Problems 11

Return by Monday 7th April

1. We define $\sigma(n)$ for $n \in \mathbb{N}$ to be the sum of the divisors of n,

$$\sigma(n) = \sum_{m|n} m.$$

(i) Prove that σ is a multiplicative function.
(ii) Evaluate $\sigma(1050)$.
(iii) Prove that

$$\sum_{m|n} \phi(m)\sigma(n/m) = nd(n).$$

(iv) Show that if $\sigma(n)$ is odd, then n is a square or twice a square.
(v) Prove that

$$\sum_{m|n} \mu(m)\sigma(n/m) = n.$$

(vi) Prove that

$$\sum_{m|n} \mu(n/m) \sum_{l|m} \mu(l)\sigma(m/l) = \phi(n).$$

2. (cf Hille (1937)) Suppose that $f(x)$ and $F(x)$ are complex-valued functions defined on $[1, \infty)$. Prove that

$$F(x) = \sum_{n \le x} f(x/n)$$

for all x if and only if

$$f(x) = \sum_{n \le x} \mu(n)F(x/n)$$

for all x.

3. Show for each positive integer k that there is a unique arithmetic function ϕ_k such that $\sum_{m|n} \phi_k(m) = n^k$. Obtain a formula for $\phi_k(n)$ and show that $\phi_k(n)$ is multiplicative.

4. Suppose that the arithmetical function $\eta(n)$ satisfies $\sum_{m|n} \eta(m) = \phi(n)$. Show that $\eta(n)$ is multiplicative and evaluate $\eta(p^k)$.

5. Evaluate $h(n) = \sum_{m|n} (-1)^m \mu(n/m)$.