Math 465 Number Theory, Spring 2025, Problems 10
Return by Monday 31st March

1. (i) Prove that if $p \neq 2$, then $\left(\frac{-5}{p}\right)_L = 1$ iff $p \equiv 1, 3, 7$ or $9 \pmod{20}$.

(ii) List those $n \le 25$ for which $x^2 + 5y^2 = n$ is soluble in integers x and y. Are there any primes of the form $p \equiv 1, 3, 7$ or $9 \pmod{20}$ for which $x^2 + 5y^2 = p$ is insoluble in x and y? Which of them are represented by $2x^2 + 2xy + 3y^2$?

2. Prove that the number

$$n = 2^k \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 1$$

cannot be represented by the sum of fewer than

$$2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2$$

positive k-th powers.

3. (i) Show that

$$\sum_{\ell | (m,n)} \mu(\ell)$$

is 1 when $(m, n) = 1$ and is 0 otherwise.

(ii) Prove that

$$\sum_{m=1, (m,n)=1}^{n} m = \frac{1}{2} n \phi(n) \quad \text{when} \quad n > 1.$$

(iii) Let $N(n, h)$ denote the number of m with $1 \le m \le n$ and $\big(m(m+h), n\big) = 1$. Prove that

$$N(n, h) = n \sum_{\ell | n} \frac{\mu(\ell)}{\ell} \rho(\ell)$$

where $\rho(\ell)$ is the number of m with $1 \le m \le \ell$ and $\ell | m(m + h)$.

(iv) Prove that $\rho(\ell)$ is multiplicative.

(v) Evaluate $\rho(p)$ and deduce that

$$N(n, h) = \phi(n) \prod_{p | n, p \nmid h} \frac{p - 2}{p - 1}.$$

4. Show that if n is a natural number, then

$$\prod_{m | n} m = n^{\frac{1}{2} d(n)}.$$