MATH 465 NUMBER THEORY, SPRING 2025, PROBLEMS 8

Return by Monday 17th March

1. Evaluate the following Legendre symbols.

$$\text{(i)} \quad \left(\frac{2}{127}\right)_L, \quad \text{(ii)} \quad \left(\frac{-1}{127}\right)_L, \quad \text{(iii)} \quad \left(\frac{5}{127}\right)_L, \quad \text{(iv)} \quad \left(\frac{11}{127}\right)_L.$$

2. Given that 5003 is prime, determine the solubility of $x^2 \equiv 2021 \pmod{5003}$.

3. (i) Prove that 3 is a QR modulo p when $p \equiv \pm 1 \pmod{12}$ and is a QNR when $p \equiv \pm 5 \pmod{12}$.

(ii) Prove that -3 is a QR modulo p for primes p with $p \equiv 1 \pmod 6$ and is a QNR for primes $p \equiv -1 \pmod 6$.

(iii) By considering $4x^2 + 3$ show that there are infinitely many primes in the residue class $1 \pmod 6$.

4. Show that for every prime p the congruence

$$x^6 - 11x^4 + 36x^2 - 36 \equiv 0 \pmod p$$

is always soluble.

5. Decide the solubility of
(i) $x^2 \equiv 219 \pmod{383}$,
(ii) $x^2 \equiv 226 \pmod{562}$,
(iii) $x^2 \equiv 429 \pmod{563}$,
(iv) $x^2 \equiv 105 \pmod{317}$.