MATH 465 NUMBER THEORY, SPRING 2025, PROBLEMS 5

Return by Monday 17th February

1. Solve the simultaneous congruences

$$\begin{aligned} x &\equiv 3 \pmod{6} \\ x &\equiv 5 \pmod{35} \\ x &\equiv 7 \pmod{143} \\ x &\equiv 11 \pmod{323} \end{aligned}$$

2. Prove that if p is an odd prime and $0 < k < p$, then (assuming $0! = 1$) $(p-k)!(k-1)! \equiv (-1)^k \pmod{p}$.

3. (i) Let $m \in \mathbb{N}$. Prove that

$$(y-1)(y^{m-1} + y^{m-2} + \cdots + y + 1) = y^m - 1.$$

(ii) Let $n \in \mathbb{N}$. Prove that

$$(x^2+1)(x^2-1)(x^{4n-4} + x^{4n-8} + \cdots + x^4 + 1) = x^{4n} - 1.$$

(iii) Let p be a prime number with $p \equiv 1 \pmod{4}$. Prove that $x^2 \equiv -1 \pmod{p}$ has exactly two solutions.

4. Prove that if a has order 3 modulo a prime p, then $1 + a + a^2 \equiv 0 \pmod{p}$, and $1 + a$ has order 6.