MATH 465 NUMBER THEORY, SPRING TERM 2025, PROBLEMS 3

Return by Monday 3rd February

1. Show that if $ad - bc = \pm 1$, then $(a + b, c + d) = 1$.

2. Suppose that $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Prove that there are integers a, b such that $(a, b) = m$ and $[a, b] = n$ if and only if $m|n$.

3. Find $(1819, 3587)$, and obtain the complete solution in integers x and y to $1819x + 3587y = (1819, 3587)$.

4. Let $\{F_n : n = 1, 2, \ldots\}$ be the Fibonacci sequence defined by $F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ and let

$$\theta = \frac{1 + \sqrt{5}}{2} = 1.6180339887498948482045868343656\ldots.$$

(i) Prove that

$$F_n = \frac{\theta^n - (-\theta)^{-n}}{\sqrt{5}}.$$

(ii) Suppose that a and b are positive integers with $b < a$ and we adopt the notation used in the description of Euclid's algorithm. Prove that for $k = 1, \ldots, s - 1$ we have $F_k \le r_{s-1-k}$ and

$$s \le 1 + \frac{\log 2b\sqrt{5}}{\log \theta}.$$

This shows that Euclid's algorithm runs in time at most linear in the bit size of $\min(a, b)$.