MATH 465 NUMBER THEORY, SPRING TERM 2025, PROBLEMS 2

Return by Monday 27th January

1. Prove that if $2^m + 1$ is an odd prime, then there is an $n \in \mathbb{N}$ such that $m = 2^n$. These are the Fermat primes. Fermat thought that all numbers of the form $2^{2^n} + 1$ are prime. Show that $641 | 2^{2^5} + 1$.

2. Let $n_1, n_2, \ldots, n_s \in \mathbb{Z}$. Define the greatest common divisor d of $n_1, n_2, \ldots, n_s$ and prove that there exist integers $m_1, m_2, \ldots, m_s$ such that $n_1 m_1 + n_2 m_2 + \cdots + n_s m_s = d$.

3. Let $a \in \mathbb{N}$ and $b \in \mathbb{Z}$. Prove that the equations $(x, y) = a$ and $xy = b$ can be solved simultaneously in integers x and y if and only if $a^2 | b$.

4. Find integers x and y such that $16801x + 2024y = (16801, 2024)$.