MATH 465 NUMBER THEORY, SPRING TERM 2025, PROBLEMS 1

Divisibility and Factorisation

Return by Wednesday 22nd January

1. Let $a, b, c \in \mathbb{Z}$. Prove each of the following.
   (i) If $a|b$ and $b|c$, then $a|c$.
   (ii) If $a|b$, then $ac|bc$.
   (iii) If $ac|bc$ and $c \neq 0$, then $a|b$.
   (iv) If $a|b$ and $a|c$, then $a|bx + cy$ for all $x, y \in \mathbb{Z}$.

2. Define the integer sequence a_n, $n \in \mathbb{N}$ by $a_1 = 1 = a_2 = 1$ and

$$a_{n+2} = 3a_{n+1} - a_n.$$

Prove that if $m|a_{n+1}$ and $m|a_n$ for some $n \in \mathbb{N}$, then $m = 1$.

3. Prove that for every $n \in \mathbb{Z}$ we have $3|n^3 - n$.

4. (i) Show that if $4|m - 1$ and $4|n - 1$, then $4|mn - 1$.
   (ii) Show that if $m, n \in \mathbb{N}$, and $4|mn + 1$, then either $4|m + 1$ or $4|n + 1$.
   (iii) Show that if $4|n + 1$, then there is a prime number p with $p|n$ and $4|p + 1$.
   (iv) Show that there are infinitely many primes p such that $4|p + 1$.