MATH 465 NUMBER THEORY, SPRING TERM 2025, PRACTISE FINAL

Note: The Final Exam for Math 465 will be Monday 5th May, 4:40pm to 6:30pm in room 312 Boucke.

1. Prove that if l, m and n are non-zero integers with $\text{GCD}(l, m) = 1$ and $l|mn$, then $l|n$.

2. Find $(1745, 1485)$ and integers x and y such that $1745x + 1485y = (1745, 1485)$.

3. Solve the simultaneous congruences $x \equiv 3 \pmod 4$, $x \equiv 2 \pmod 7$, $x \equiv 7 \pmod 9$

4. Find all solutions to the congruence $9x^{58} + 4x^{30} + 2x \equiv 0 \pmod{29}$.

5. Find a primitive root modulo 17 and draw up a table of discrete logarithms to this base. Hence, or otherwise, find all solutions to the following congruences.
(i) $x^{16} \equiv 3 \pmod{17}$,
(ii) $x^{21} \equiv 3 \pmod{17}$,
(iii) $x^{30} \equiv 8 \pmod{17}$.

6. Evaluate the following Legendre symbols, showing your working.

$$\text{(i)} \quad \left(\frac{-1}{103}\right)_L, \quad \text{(ii)} \quad \left(\frac{2}{103}\right)_L, \quad \text{(iii)} \quad \left(\frac{7}{103}\right)_L, \quad \text{(iv)} \quad \left(\frac{83}{103}\right)_L.$$

7. (i) Prove that -6 is a quadratic non-residue modulo 13 (ii) Prove that if $2x^2 + 3y^2 \neq 0$ and $13|2x^2 + 3y^2$, then 13 divides $2x^2 + 3y^2$ exactly to an even power. (iii) Find all solutions to the diophantine equation $2x^2 + 3y^2 = 26z^2 + 39t^2$.

8. Let $\sigma(n)$ denote the sum of the divisors of n, $\sigma(n) = \sum_{m|n} m$.
(i) When $X \geq 1$, prove that

$$\sum_{n \leq X} \frac{\sigma(n)}{n} = \sum_{m \leq X} \sum_{l \leq X/m} \frac{1}{l}.$$

(ii) Prove that

$$\sum_{n \leq X} \frac{\sigma(n)}{n} = \sum_{l \leq X} \frac{1}{l} \left\lfloor \frac{X}{l} \right\rfloor.$$

(iii) Prove that

$$\sum_{n \leq X} \frac{\sigma(n)}{n} = CX + O(\log X)$$

where $C = \sum_{l=1}^{\infty} l^{-2}$.