MATH 465 SPRING TERM 2025, PRACTICE EXAM 1

Note: Exam 1 will be 1:25–2:15, Wednesday 5th February in 216 Thomas

1. Suppose that $l, m, n \in \mathbb{N}$. Prove that $(lm, ln) = l(m, n)$.

Suppose that m, n, l have the canonical decompositions $m = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, $n = p_1^{\beta_1} \dots p_s^{\beta_s}$, $l = p_1^{\gamma_1} \dots p_s^{\gamma_s}$ where the α_j, β_j, γ_j are non–negative integers. Then $(lm, ln) = \prod_j p_j^{\min(\gamma_j+\alpha_j, \gamma_j+\beta_j)}$ and $l(m, n) = \prod_j p_j^{\gamma_j+\min(\alpha_j,\beta_j)}$ and the result follows on observing that $\min(\gamma + \alpha, \gamma + \beta) = \gamma + \min(\alpha, \beta)$.

2. (i) Show that if $(l, 6) = 1$, then $6|l \pm 1$. (ii) Show that if $6|l - 1$ and $6|m - 1$, then $6|lm - 1$. (iii) Show that if $6|lm + 1 \pmod 6$, then either $6|l + 1$ or $6|m + 1$. (iv) Show that if $n \in \mathbb{N}$ and $6|n + 1$, then there is a prime number p such that $p|n$ and $6|p + 1$. (v) Show that there are infinitely many primes of the form $6k - 1$.

(i) Write $l = 6m+k$ where $0 \le k \le 5$. Then $(l, 6) = (k, 6) = 1$ iff $k = 1$ or 5. Thus $l \equiv \pm 1 \pmod 6$. (ii) At once by Theorem 3.2. (iii) We have $lm \equiv -1 \pmod 6$. Hence $(lm, 6) = 1$, so $(l, 6) = (m, 6) = 1$. Thus, by (i). $l \equiv \pm 1 \pmod 6$ and $m \equiv \pm 1 \pmod 6$. But if $l \equiv m \equiv 1 \pmod 6$, then by (ii), we would have $lm \equiv 1 \pmod 6$. Hence at least one of l, m must lie in the residue class -1 modulo 6. (iv) Write $n = p_1 \dots p_s$ where the primes are not necessarily distinct. Then by (iii) either $p_1 \dots p_{s-1} \equiv -1 \pmod 6$ or $p_s \equiv -1 \pmod 6$. By repeated application of this argument it follows that at least one of the p_j lies in the residue class -1 modulo 6. (v) Suppose there are only finitely many, say $p_1, \dots , p_s$. Put $k = p_1 \dots p_s$. Then by (iv) there is a p of the same form which divides $6k - 1$. But it also divides k so it divides $6k - (6k - 1) = 1$ which is impossible.

3. Find all pairs of integers x and y such that $922x + 2163y = 7$.

First solve $922x_1 + 2163y_1 = (922, 2163)$ (1). $2163 = 2.922 + 319$, $922 = 2.319 + 284$, $319 = 284 + 35$, $284 = 8.35 + 4$, $35 = 8.4 + 3$, $4 = 3 + 1$. Hence $(922, 2163) = 1 = 4 - 3 = 4 - (35 - 8.4) = 9(284 - 8.35) - 35 = 9.284 - 73(319 - 284) = 82(922 - 2.319) - 73.319 = 82.922 - 237(2163 - 2.922) = 556.922 - 237.2163$. Thus $x_1 = 556$, $y_1 = -237$ is a solution of (1). Hence $x = 7x_1 = 3892$, $y = -7.237 = -1659$ is a solution to the question. Therefore the general solution is $x = 3892 + 2163t$, $y = -1659 - 922t$.

4. (i) Prove that if $x \in \mathbb{Z}$, then $4|x^2$ or $4|x^2 - 1$. (ii) Prove that $5y^2 + 2 = z^2$ has no solutions with $y, z \in \mathbb{Z}$.

(i) We only have to test the residue classes $0^2, 1^2, 2^2, 3^2$ and they are $0, 1, 0, 1$ respectively. (ii) $5y^2 + 2$ lies in the residue classes 2 or $5 + 2 \equiv 3$. z^2 lies in 0 or 1.