MATH 421 COMPLEX ANALYSIS, FALL TERM 2004, PRACTICE FINAL EXAM SOLUTIONS

1. Set of points equidistant from $-i$ and $3i$, so the straight line thro' i parallel to the real axis. (ii) Circle centre $17/15$, radius $8/15$.
2. Show that $R\backslash\{z\}$ is open and polygonally connected. Let $a \in R\backslash\{z\}$. Then $a \in R$ and $a \neq z$. Hence there is a disc $D(a;\delta)$ in R with $0 < \delta < |z-a|$ and this disc is also in $R\backslash\{z\}$. Now let $a, b \in R\backslash\{z\}$ with $a \neq b$. Then $a, b \in R$ and there is a polygon $[w_0, \ldots, w_k]$ in R with $w_0 = a$, $w_k = b$. If z not on this polygon then we are done. If it is consider a square with its centre at z and of sufficiently small side length that it is in R and intersects the polygon and use the sides of the square to detour around z.
3. $u = e^{-y}\cos x$, $v = -e^{-y}\sin x$. Thus $u_x = -e^{-y}\sin x$, $v_y = e^{y}\sin x$, $u_y = -e^{-y}\cos x$, $v_x = -e^{-y}\cos x$, $-e^{-y}\sin x = e^{-y}\sin x$, $\sin x = 0$, $-e^{-y}\cos x = e^{-y}\cos x$, $\cos x = 0$. But $\cos^2 x + \sin^2 x = 1$.
4. Let $F(z) = \exp(z + 1/z)$. Then F is differentiable on $\mathcal{C}$ and $F'(z) = (1 - z^{-2})\exp(z+1/z)$ and so F acts as a primitive for the integrand on $\mathcal{C}$. Hence the integral is $F(1) - F(-1) = e^2 - e^{-2} = 2\sinh 2$.
5. (i) $\mathbb{C}\backslash\{i, -i\}$. (ii) The integrand is $f(z)(z-i)^{-2}$ where $f(z) = (z+i)^{-2}$ is analytic on and inside $\mathcal{C}$. Hence, by the Cauchy integral formula, the integral is $2\pi i f'(i) = 2\pi i(-2)(2i)^{-3} = \pi/2$.
6. $e^z = e^{-1}e^{z+1} = \sum_{k=0}^{\infty} \frac{e^{-1}}{k!}(z+1)^k$, valid for all $z \in \mathbb{C}$. Hence $(z+1)^{-2}e^z = \sum_{n=-2}^{\infty} \frac{e^{-1}}{(n+2)!}(z+1)^n$, valid for all $z \neq -1$. This, by the identity theorem for Laurent series this is the Laurent expansion about 0 and the residue at -1 is e^{-1}.
7. The integral is $\frac{1}{2}\Im \lim_{R\to\infty} I_R$ where $I_R = \int_{-R}^{R} f(z)dz$ where $f(z) = \frac{ze^{iz}}{z^2+4}$. We suppose R and T are large and positive and define L_1, L_2 and L_3 to be the line segments joining R to $R+iT$, $R+iT$ to $-R+iT$, and $-R+iT$ to $-R$. On L_1 and L_3, $z = \pm R + it$, $|f(z)| \leq 2e^{-t}R^{-1}$. Thus $|\int_{L_j} f(z)dz| \leq 2R^{-1}\int_0^T e^{-t}dt < 2R^{-1}$ $(j = 1, 3)$. On L_2, $z = x + iT$, so $|f(z)| \leq 2T^{-1}e^{-T}$ and $|\int_{L_2} f(z)dz| \leq 4RT^{-1}e^{-T}$. The residue of f at at $2i$ is $\frac{2ie^{-2}}{4i} = \frac{1}{2}e^{-2}$. Hence, by the residue theorem, $I_R + \int_{L_1+L_2+L_3} f(z)dz = \pi i e^{-2}$. Thus $|I_R - \pi i e^{-2}| \leq 4R^{-1} + 4RT^{-1}e^{-T}$. This holds for all large T, so $|I_R - \pi i e^{-2}| \leq 4R^{-1}$. Let $R \to \infty$. Then our integral is $\frac{1}{2}\pi e^{-2}$.