MATH 421 COMPLEX ANALYSIS, FALL TERM 2004, PRACTICE EXAM 2 SOLUTIONS

The second exam is on Wednesday 10th November, at 9:05 in 109 Bouke.

1. (i) Show that $f'(z)$ exists at no point of $\mathbb{C}$ when (i) $f(x+iy) = x^2 - y^2 - x + i(2xy + y)$. $u_1 = 2x - 1$, $v_2 = 2x + 1$, $u_1 - v_2 = -2 \neq 0$. (ii) $f(z) = e^{\overline{z}}$. $e^{\overline{z}} = e^x(\cos y - i \sin y)$. $u_1 = e^x \cos y$, $v_2 = -e^x \cos y$, $u_2 = -e^x \sin y$, $v_1 = -e^x \sin y$. $u_1 = v_2$ iff $2^x = \cos y = 0$ iff $\cos y = 0$. $u_2 = -v_1$ iff $2e^x \sin y = 0$ iff $\sin y = 0$. But $\sin^2 y + \cos^2 y = 1$, so $\cos y$ and $\sin y$ cannot be simultaneously 0.

2. Use the Cauchy-Riemann equations to show in each case that $f(z)$ and $f'(z)$ are entire. (i) $f(x+iy) = -2xy + i(x^2 - y^2)$. $u_1 = v_2 = -2y$, $u_2 = -v_1 = -2x$, and all are continuous. Thus $f'(x+iy)$ exists and equals $-2y + 2ix$. Now $u_1 = v_2 = 0$, $u_2 = -v_1 = -2$ and all are continuous. (ii) $f(x+iy) = \sinh x \cos y + i \cosh x \sin y$. $u_1 = v_2 = \cosh x \cos y$, $u_2 = -v_1 = -\sinh x \sin y$ and all are continuous. Hence $f'(x+iy) = \cosh x \cos y + i \sinh x \sin y$. Now $u_1 = v_2 = \sinh x \cos y$, $u_2 = -v_1 = -\cosh x \sin y$ and all are continuous.

3. In each case what is the largest domain of holomorphicity of the given function? (i) $f(z) = \frac{1}{z}$, (ii) $f(z) = \frac{1}{z^2+1}$, (iii) $f(z) = \log(z^2+1)$, where we take the branch of the logarithm with $-\pi < \Im \log w \leq \pi$. (i) $\mathbb{C}\backslash\{0\}$. (ii) $\mathbb{C}\backslash\{-i, i\}$. (iii) $\mathbb{C}\backslash\{iy : y \geq 1 \text{ or } y \leq -1\}$.

4. Let $\mathcal{C}$ denote the path $\mathcal{C} = \{z(t) : 0 \leq t \leq 2\}$ where $z(t) = t$ $(0 \leq t \leq 1)$, $z(t) = 1 + i(t-1)$ $(1 \leq t \leq 2)$. Evaluate (i) $\int_{\mathcal{C}} z^3 dz$, (ii) $\int_{\mathcal{C}} e^z dz$. (i) $\int_{\mathcal{C}} z^3 dz = \int_0^1 t^3 dt + \int_1^2 (1 + i(t-1))^3 i dt = -1$. (ii) $\int_{\mathcal{C}} e^z dz = \int_0^1 e^t dt + \int_1^2 e^{1+i(t-1)} i dt = e^{1+i} - 1$.