Continuity

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- **Example 8.1.** *Suppose that* $c_0, c_1, \ldots, c_m \in \mathbb{R}$, $\xi \in \mathbb{R}$ *and*

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- *Then by the combination theorem applied many times it follows that P is continuous at ξ.*
- *In other words every polynomial is continuous at every real number x.*
- *More generally, it follows from the combination theorem that every rational function $\frac{P(x)}{Q(x)}$ is continuous at every real number ξ for which $Q(\xi) \neq 0$.*

- **Example 8.2.** *The function* $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} x + \frac{1}{2} & -\frac{1}{2} \leq x < 0, \\ x - \frac{1}{2} & 0 \leq x \leq \frac{1}{2}. \end{cases}$$

is discontinuous at 0*, but the function* $g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ *defined by*

$$g(x) = \begin{cases} (x + \frac{1}{2})^2 & -\frac{1}{2} \leq x < 0, \\ (x - \frac{1}{2})^2 & 0 \leq x \leq \frac{1}{2}. \end{cases}$$

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- *Then we say that* f *is* **continuous from below, or from the left, at** ξ *when* $f(x) \to f(\xi)$ *as* $x \to \xi-$*. Otherwise we say that* f *is discontinuous from the left at* ξ*.*

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- **Example 8.3.** *In Example 8.2. the function*

$$f(x) = \begin{cases} x + \frac{1}{2} & -\frac{1}{2} \le x < 0, \\ x - \frac{1}{2} & 0 \le x \le \frac{1}{2}. \end{cases}$$

is discontinuous from the left at 0, *but continuous from the right at* 0.

- For continuity at a point we can apply Chapter 7.

Theorem 1 (Comb. Theorem for Pointwise Continuity)

Suppose that $a < \xi < b$, $f, g : (a, b) \mapsto \mathbb{R}$, and $f(x)$ and $g(x)$ are continuous at ξ, and that $\lambda, \mu \in \mathbb{R}$. Then
(i) $\lambda f(x) + \mu g(x)$ is continuous at ξ,
(ii) $f(x)g(x)$ is continuous at ξ,
(iii)if $g(\xi) \neq 0$, then $\dfrac{f(x)}{g(x)}$ is continuous at ξ.

Theorem 2 (Comb. Theorem for Left and Right Continuity)

Suppose that $a < \xi$, $f, g : (a, \xi] \mapsto \mathbb{R}$, and $f(x)$ and $g(x)$ are continuous from below at ξ, and that $\lambda, \mu \in \mathbb{R}$. Then
(i) $\lambda f(x) + \mu g(x)$ is continuous from below at ξ,
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(iii)if $g(\xi) \neq 0$, then $\dfrac{f(x)}{g(x)}$ is continuous from below at ξ.
There are corresponding statements for continuity from above when f and g are defined on $[\xi, b)$.

- The following is also useful.

Theorem 3

Suppose $a < b$, $\xi \in (a, b)$. *Then* $f : (a, b) \mapsto \mathbb{R}$ *is continuous at* ξ *if and only if for* **every** *sequence* $\langle x_n \rangle$ *in* (a, b) *satisfying* $\lim_{n\to} x_n = \xi$ *we have*

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- Then choose N so that when $n > N$ we have $|x_n - \xi| < \delta$. Hence $|f(x_n) - f(\xi)| < \varepsilon$.

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- **"If."** Suppose f is not continuous at ξ but every sequence $\langle x_n \rangle$ in (a, b) converging to ξ satisfies

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- Take $\delta = 1/n$. Then by the assertion just above there is x, so that $|x - \xi| < \frac{1}{n}$. Take $x_n = x$. Now $|f(x_n) - f(\xi)| \geq \varepsilon_0$, but $\langle x_n \rangle$ is converging to ξ and so contradicts (1) above.

- An interesting application. **Example 8.4.** *Let* $\exp(x) : x \mapsto \mathbb{R}$ *be the function defined in (6.8). Then*
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- *Proof.* (i) Suppose that $|x| < 1$. Then

$$|\exp(x) - 1| \le \sum_{n=1}^{\infty} \frac{|x|^n}{n!} \le \sum_{n=1}^{\infty} |x|^n$$
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- Let $\varepsilon > 0$ and choose $\delta = \frac{\varepsilon}{1+\varepsilon}$. Thus, when $|x| < \delta$ we have

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- (ii) We have $\exp(\xi + h) = \exp(\xi)\exp(h) \to \exp(\xi)$ as $h \to 0$ by (i), i.e. $\exp(x) \to \exp(\xi)$ as $x \to \xi$.

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- *Suppose instead that* I *is the* **closed interval** $[a, b]$ *and* $f : I \mapsto \mathbb{R}$. *Then* f *is continuous on* I *when it is continuous at every point* $\xi \in (a, b)$, *continuous from the right at* a *and continuous from the left at* b.

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- *When* f *is continuous at every* $\xi \in \mathbb{R}$ *then we say that* f *is continuous on* $\mathbb{R}$.

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- **Example 8.6.** *Let* $c_0, c_1, \ldots, c_n \in \mathbb{R}$ *and* $P : \mathbb{R} \mapsto \mathbb{R} : P(x) = c_0 + c_1 x + \cdots + c_n x^n$. *Then* P *is continuous on* $\mathbb{R}$.

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- **Example 8.7** *The functions* $\exp(x)$, $\cos(x)$ *and* $\sin(x)$ *are continuous on* $\mathbb{R}$.

- Now we can apply the combination theorem for continuity at a point.

Theorem 4 (Combination Theorem for Continuity on an Interval)

Suppose that $a < b$ and $I = (a, b)$ or $[a, b]$, $f, g : I \mapsto \mathbb{R}$, and $f(x)$ and $g(x)$ are continuous on I. Suppose further that $\lambda, \mu \in \mathbb{R}$. Then
(i) $\lambda f(x) + \mu g(x)$ is continuous on I,
(ii) $f(x)g(x)$ is continuous on I,
(iii)if $g(x) \neq 0$ for $x \in I$, then

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Suppose that $a < b$ and $f : [a, b] \mapsto \mathbb{R}$ is continuous on $[a, b]$. Then f is bounded.

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- *Proof.* Suppose that $f([a, b])$ is unbounded above. (If instead unbounded below replace f by $-f$.) Then given any $n \in \mathbb{N}$ there is an $x_n \in [a, b]$ such that $f(x_n) > n$.

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- But $a \le x_n \le b$, so $\langle x_n \rangle$ is bounded and by Bolzano Weierstrass it has a convergent subsequence $\langle x_{m_n} \rangle$.

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- But $a \le x_n \le b$, so $\langle x_n \rangle$ is bounded and by Bolzano Weierstrass it has a convergent subsequence $\langle x_{m_n} \rangle$.
- Let $\ell = \lim_{n\to\infty} x_{m_n}$. Then $a \le \ell \le b$, and since function f is continuous at ℓ there is a $\delta > 0$ so that when $|x_{m_n} - \ell| < \delta$ we have $|f(x_{m_n}) - f(\ell)| < 1$.

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- Let $\ell = \lim_{n\to\infty} x_{m_n}$. Then $a \le \ell \le b$, and since function f is continuous at ℓ there is a $\delta > 0$ so that when $|x_{m_n} - \ell| < \delta$ we have $|f(x_{m_n}) - f(\ell)| < 1$.
- By the triangle inequality, for all $n \in \mathbb{N}$, $n \le m_n <$

$$f(x_{m_n}) = (f(x_{m_n}) - \ell) + \ell \le |f(x_{m_n}) - \ell| + |\ell| < 1 + |\ell|$$

contradicting the Archimedean property of $\mathbb{N}$.

- This leads to the following remarkable and very useful result.

Theorem 6

Suppose that $a < b$ *and* f *is continuous on* $[a, b]$. *Then* f *attains its bounds. In other words there are* $\eta, \xi \in [a, b]$ *such that* $f(\eta) = \inf f([a, b])$ *and* $f(\xi) = \sup f([a, b])$.

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Suppose that $a < b$ and f is continuous on $[a, b]$. Then f attains its bounds. In other words there are $\eta, \xi \in [a, b]$ such that $f(\eta) = \inf f([a, b])$ and $f(\xi) = \sup f([a, b])$.

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- Hence $0 < \frac{1}{B} < \Lambda - f(x)$ and so $f(x) < \Lambda - \frac{1}{B}$ which contradicts the definition of Λ.

- We now come to a theorem which is used all the time in applications. It is especially important in that it underpins all zero finding techniques for continuous functions.

Theorem 7 (The Intermediate Value Theorem)

Suppose that $a < b$, $f : [a, b] \mapsto \mathbb{R}$ *is continuous on* $[a, b]$ *and*

$$\inf f([a, b]) \le \lambda \le \sup f([a, b]).$$

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- **Remark.** *This theorem says in effect that, when f is continuous on a closed interval $[a, b]$, the set $f([a, b])$ is also an interval.*

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- In any case we have (1), (2), (3), (4) with n replaced by $n + 1$.

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- Therefore

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Corollary 8

The image of exp *is* $\mathbb{R}^+$

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- Here are some examples connected with the last few theorems.
- **Example 8.8.** *Let* $f(x) = x^2 - x$ *be defined on* $I = \left[-\frac{1}{2}, \frac{3}{4}\right]$. *Then*

$$\inf f(I) = -\tfrac{1}{4},\ f\left(\tfrac{1}{2}\right) = -\tfrac{1}{4},$$

$$\sup f(I) = \tfrac{3}{4},\ f\left(-\tfrac{1}{2}\right) = \tfrac{3}{4},$$

$$-\tfrac{1}{4} < \tfrac{5}{16} < \tfrac{3}{4}, f\left(-\tfrac{1}{4}\right) = \tfrac{5}{16}.$$

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- *Proof.* For brevity write $f(x) = x^3 - 3x^2 + 1$.
- Then

$$f(-1) = -3,\ f(0) = 1,\ f(1) = -1,\ f(3) = 1$$

  and f is continuous on each of the intervals $[-1, 0]$, $[0, 1]$, $[1, 3]$.
- Hence there are ξ_1, ξ_2, ξ_3 so that

$$-1 < \xi_1 < 0 < \xi_2 < 1 < \xi_3 < 3$$

  and

$$f(\xi_1) = f(\xi_2) = f(\xi_3) = 0.$$

- **Example 8.10** *Prove that the curve* $y = x^2$ *intersect the curve* $y = x^3 - 2x^2 + 1$ *in three places.*
- *Proof.* At a point of intersection $x^2 = x^3 - 2x^2 + 1$, so that $x^3 - 3x^2 + 1 = 0$.
- Hence see previous example.

- **Example 8.11.** *Suppose that* f *is continuous on* $[0,1]$ *and* $f(0) = f(1)$*. Prove that there is a* $\xi \in [0,1]$ *so that*

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- If $f(0) = f(1/2)$, then we are done.
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- Hence g changes sign on $[0, 1/2]$. Thus, by the Intermediate Value Theorem there is a $\xi \in (0, 1/2)$ such that $g(\xi) = 0$ and we are done once more.

- We can now say something more about sin and cos

Theorem 9

The function cos *changes sign on the interval* $[0, 2]$*. We define* $\frac{\pi}{2}$ *to be the smallest positive zero of* cos*. Then* cos *and* sin *are periodic with period* 2π, $\sin(0) = \sin(\pi) = 0$, *and*

$$\sin\frac{\pi}{2} = 1, \sin\frac{3\pi}{2} = -1, \cos(x) = \sin\left(\frac{\pi}{2} - x\right).$$

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- *Proof* By the definition of cos, (6.10), $\cos(0) = 1$ and

$$\cos(2) = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \sum_{k=2}^{\infty} \frac{2^{4k-2}}{(4k-2)!}\left(1 - \tfrac{2^2}{(4k-1)4k}\right)$$
$$< 1 - 2 + \tfrac{2}{3} = -\tfrac{1}{3}.$$

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- By continuity, $\cos(\varpi)=0$ and, as $\cos(0)=1$, $\varpi>0$.
- Define $\pi=2\varpi$.

- For any non-negative integer k, when $0 < x \le 2$ we have

$$\frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+3}}{(4k+3)!} = \frac{x^{4k+1}}{(4k+1)!}\left(1 - \frac{x^2}{(4k+2)(4k+3)}\right) > 0.$$

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- Hence, by the definition of sin, (6.9), we have $\sin(\varpi) > 0$.

- By the addition formulæ Exercise 6.5.1, we have

$$
\begin{aligned}
\sin(\pi) &= 2\sin(\varpi)\cos(\varpi) = 0 \\
\cos(\pi) &= 2(\cos(\varpi))^2 - 1 = -1, \\
\cos(2\pi) &= 1 - 2(\sin(\pi))^2 = 1, \\
\sin(2\pi) &= 2\sin(\pi)\cos(\pi) = 0, \\
\sin(x + 2\pi) &= \sin(x)\cos(2\pi) + \cos(x)\sin(2\pi) = \sin(x), \\
\cos(x + 2\pi) &= \cos(x)\cos(2\pi) - \sin(x)\sin(2\pi) = \cos(x), \\
-1 = \cos(\pi) &= 1 - 2\sin^2(\varpi), \\
\sin^2(\varpi) &= 1, \\
\sin(\varpi) &= 1, \\
\cos(-x) &= \cos(x), \\
\sin(-x) &= -\sin(x).
\end{aligned}
$$

- Thus

$$\begin{aligned}
\sin(\varpi - x) &= \sin(\varpi)\cos(-x) + \cos(\varpi)\sin(-x) \\
&= \cos(x), \\
\sin(3\varpi) &= \sin(\varpi + \pi) \\
&= \cos(-\pi) \\
&= \cos(\pi) \\
&= -1.
\end{aligned}$$

- Consider a real valued function defined on some domain $\mathcal{D} \in \mathbb{R}$, $f : \mathcal{D} \mapsto \mathbb{R}$. Then the definition of continuity, Definition 8.1 is a pointwise definition, even in the special case of an interval, Definition 8.3.

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or equivalently $|\xi - x| < \varepsilon x \xi < \varepsilon\xi(x - \xi) + \varepsilon\xi^2$.

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- *Now* δ *cannot be taken to be independent of* ξ, *for taking* ξ *arbitrarily close to* 0 *would contradict* $\delta > 0$.

Uniform Continuity

- When we have a situation in which it is possible to find a universal δ it is usual to associate the word *uniform* with it.
- **Definition 8.4.** *Suppose that $\mathcal{S} \subset \mathbb{R}$ and $f : \mathcal{S} \mapsto \mathbb{R}$ has the property that for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $x, y \in \mathcal{S}$ and $|x - y| < \delta$ we have*

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- *An equivalent statement is that for every $\varepsilon > 0$ there is $\delta > 0$ such that*

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- The following contrasts open and closed intervals.

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- $\langle x_n \rangle$ is in $[a, b]$ and so by B-W it has a convergent sub sequence $\langle x_{m_n} \rangle$. Then $\langle y_{m_n} \rangle$ has a convergent subsequence $\langle y_{m_{k_n}} \rangle$. Moreover $|x_{m_{k_n}} - y_{m_{k_n}}| < \dfrac{1}{m_{k_n}} \to 0$ as $n \to \infty$.

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- They have a common limit, $\ell \in [a, b]$ so by the continuity of f at ℓ and Theorem 8.3 $0 = \lim_{n\to} |f(x_{m_{k_n}}) - f(y_{m_{k_n}})| \geq \varepsilon_0$ which gives the required contradiction.