# Limits of Functions

Robert C. Vaughan

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- *When $\mathcal{S} = \mathcal{A}$ we call $f(\mathcal{A})$ the* **image or range of** *$f$.*
- *The set $\mathcal{B}$, which may have elements which are not in $f(\mathcal{A})$ is called the* **codomain of** *$f$. We can also think of the function $f$ as being the set of ordered pairs $(x, y)$ in which $x$ and $y$ are connected by the rule $y = f(x)$.*

- *When no element $y$ of the codomain appears in more than one ordered pair, then the function is called* **bijective***, which means that to each point in the image there is a unique member of the domain, i.e. there is an* **inverse function** $f^{-1}(y) = x$ *with the property that* $f^{-1}(f(x)) = x$ *and* $f(f^{-1}(y)) = y$*.*

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- **Example 7.2.** *The equation* $y^2 = x$ *with* $x \in \mathbb{R}$ *and* $y \in \mathbb{R}$ **does not** *define a function from* $\mathbb{R}$ *to* $\mathbb{R}$ *because given* $x > 0$ *there are two values of* $y$ *for which this holds.*

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- *However if we take* $\mathcal{A} = \{x : x \geq 0\}$, $\mathcal{B} = \{y : y \geq 0\}$, *then the equation* $y^2 = x$ **does** *define a function because given* $x \in \mathcal{A}$ *there is only one corresponding* $y \in \mathcal{B}$.

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- **Definition 7.2.** *Suppose that the function* $f$ *is defined on a subset* $\mathcal{S}$ *of* $\mathbb{R}$ *and its codomain is* $\mathbb{R}$. *Then we say that* $f$ *is* **bounded above by** $H$ *when the image* $f(\mathcal{S})$ *is bounded above by* $H$. *Likewise we define* **bounded below by** $h$ *when the image is bounded below by* $h$, *and* **bounded** *when it is both bounded above and below.*

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- *Likewise when* $f(\mathcal{S})$ *is bounded below we use the corresponding term* **minimum** *for infima which are attained.*
- **Example 7.3.** *The function* $f : (0,1] \mapsto \mathbb{R} : f(x) = \frac{1}{x}$ *is unbounded, but it is bounded below and* $\inf f((0,1]) = 1$*, so it has minimum* $1$ *which is attained with* $x = 1$*.*

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- *2. When $f(x_1) < f(x_2)$ for every pair $x_1, x_2$ with $x_1 < x_2$ we call it* **strictly increasing***, and on the other hand when $f(x_1) > f(x_2)$ for every pair $x_1, x_2$ with $x_1 < x_2$ we call it* **strictly decreasing***.*

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- *3. Such functions are called* **monotonic** *in case 1. and* **strictly monotonic** *in case 2.*
- *4. With reference to the last paragraph of Definition 7.1. it follows that every strictly monotonic function has an inverse from its image.*

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- In view of 4. above it follows that exp has an inverse function.

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- *Hence* $\log(x)$ *has domain* $\mathbb{R}^+$ *and image* $\mathbb{R}$*. It also satisfies*

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*for* $x \in \mathbb{R}$ *and* $y \in \mathbb{R}^+$*.*
- *Given* $u, v$ *in the domain of* $\log$ *there will be* $x, y \in \mathbb{R}$ *so that* $x = \log u$*,* $y = \log v$ *and so* $u = \exp(x)$*,* $v = \exp(y)$*. Thus* $uv = \exp(x)\exp(y) = \exp(x + y)$ *and*

$$\log(uv) = x + y = \log(u) + \log(v).$$

*We can now use this to define, whenever* $a > 0$*,*

$$a^x : \mathbb{R} \mapsto \mathbb{R}^+ : x \mapsto \exp\big(x \log(a)\big).$$

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- We also want to avoid making any assumptions about the behaviour of $f$ at $\xi$
- Thus in the first instance given a $\xi$ we will restrict our attention to functions whose domain contains the two open intervals $(a, \xi)$ and $(\xi, b)$ where $a < \xi < b$.

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- **Definition 7.6. Limit of a function.** *Suppose that* $a < \xi < b$, $\mathcal{A} \subset \mathbb{R}$ *and* $\mathcal{B} \subset \mathbb{R}$, $f : \mathcal{A} \mapsto \mathcal{B}$ *and* $(a, \xi) \cup (\xi, b) \in \mathcal{A}$.

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- *Then*

$$\lim_{x \to \xi} f(x) = \ell,$$

*or equivalently*

$$f(x) \to \ell \text{ as } x \to \xi,$$

*means that there is an* $\ell \in \mathbb{R}$ *such that for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *so that whenever* $x \in \mathcal{A}$ *and*

$$0 < |x - \xi| < \delta$$

*we have*

$$|f(x) - \ell| < \varepsilon.$$

- Restatement: there is an $\ell \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \in \mathcal{A}$ and

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- We should expect that, just as for $N$, when we come to find a suitable $\delta$ it depends on $\varepsilon$.
- We should also note the condition $0 < |x - \xi|$. We want to include the possibility that the limit $\ell$ differs from $f(\xi)$ if the latter should exist.

- **Example 7.5.** *Suppose that* $f : (0,1) \mapsto \mathbb{R}$ *is defined by*

$$f(x) = \begin{cases} 0 & x \neq \frac{1}{2}, \\ 1 & x = \frac{1}{2}. \end{cases}$$

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- *To see this take* $\delta = \frac{1}{2}$ *in the definition.*
- *Then for* $0 < |x - \frac{1}{2}| < \delta$*, so that* $0 < x < \frac{1}{2}$ *or* $\frac{1}{2} < x < 1$ *we have*

$$|f(x) - 0| = |0 - 0| = 0 < \varepsilon.$$

- Here is a more typical example.

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- Then whenever $0 < |x - \xi| < \delta$, by the triangle inequality,

$$\begin{aligned} |f(x) - \xi^2| &= |x^2 - \xi^2| \\ &= |x - \xi||x + \xi| \\ &= |x - \xi||(x - \xi) + 2\xi| \\ &\leq |x - \xi|(|x - \xi| + 2|\xi|) \\ &< \delta(\delta + 2|\xi|) \\ &\leq \frac{\varepsilon}{1 + 2|\xi|}(1 + 2|\xi|) \\ &= \varepsilon. \end{aligned}$$

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  **Example 7.6.** *Let* $f:\mathbb{R}\mapsto\mathbb{R}: f(x)=x^2$ *and* $\xi\in\mathbb{R}$. *Then* $\lim_{x\to\xi} f(x)=\xi^2$.
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$$\begin{aligned}|f(x)-\xi^2| &= |x^2-\xi^2|\\ &= |x-\xi||x+\xi|\\ &= |x-\xi||(x-\xi)+2\xi|\\ &\le |x-\xi|(|x-\xi|+2|\xi|)\\ &< \delta(\delta+2|\xi|)\\ &\le \frac{\varepsilon}{1+2|\xi|}(1+2|\xi|)\\ &= \varepsilon.\end{aligned}$$

- See how $\delta$ has to depend on $\xi$ as well as $\varepsilon$.

- Here is an example where the limit does not exist.

**Example 7.7.** *Let* $f : (0,2) \mapsto \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} 0 & (0 \le x \le 1), \\ 1 & (1 < x < 2). \end{cases}$$

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- When $1 - \delta < x_1 < 1$ we have $f(x_1) = 0$ and when $1 < x_2 < 1 + \delta$ we have $f(x_2) = 1$.
- Hence, by the triangle inequality
  $$\begin{aligned} 1 = |f(x_2) - f(x_1)| &= |(f(x_2) - \ell) - (f(x_1) - \ell)| \\ &\le |f(x_2) - \ell| + |f(x_2) - \ell| \\ &< \frac{1}{3} + \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

- **Example 7.8.** *Let* $f : \mathbb{R} \mapsto \mathbb{R} : x \mapsto x^3 + x$. *Prove that* $\lim_{x \to 2} f(x) = 10$.

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- *Proof.* Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \frac{\varepsilon}{20}\right\}$. Then whenever $|x - 2| < \delta$ we have

$$
\begin{aligned}
|f(x) - 10| &= |x^3 + x - 10| \\
&= |(x-2)(x^2 + 2x + 5)| \\
&= |x-2||(x-2)^2 + 6(x-2) + 13| \\
&\le |x-2|(|x-2|^2 + 6|x-2| + 13) \\
&< \delta(\delta^2 + 6\delta + 13) \\
&\le \frac{\varepsilon}{20}(1^2 + 6 + 13) \\
&= \varepsilon.
\end{aligned}
$$

- As with sequences we will need to combine limits. The proofs of the next two theorems follow in the same way as those for sequences and are left as exercises.

**Theorem 1 (Combination Theorem for Functions)**

*Suppose* $a < \xi < b$, $f, g : (a,\xi) \cup (\xi, b) \mapsto \mathbb{R}$, $f(x) \to \ell$ *and* $g(x) \to m$ *as* $x \to \xi$, *and* $\lambda, \mu \in \mathbb{R}$. *Then*
*(i)* $\lambda f(x) + \mu g(x) \to \lambda\ell + \mu m$ *as* $x \to \xi$,
*(ii)* $f(x)g(x) \to \ell m$ *as* $x \to \xi$,
*(iii) and when* $m \neq 0$ *we have* $\dfrac{f(x)}{g(x)} \to \dfrac{\ell}{m}$ *as* $x \to \xi$.

**Theorem 2 (Sandwich Theorem for Functions)**

*Suppose that* $a < \xi < b$, $f, g, h : (a,\xi) \cup (\xi, b) \mapsto \mathbb{R}$,

$$g(x) \le f(x) \le h(x) \text{ when } x \in (a,\xi) \cup (\xi, b),$$

$g(x) \to \ell$ *and* $h(x) \to \ell$ *as* $x \to \xi$. *Then* $f(x) \to \ell$ *as* $x \to \xi$.

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- Typically this happens when a function is only defined on a closed interval $[a, b]$ and we want to understand the limiting behaviour at $a$ and $b$.
- It can also happen with examples like $f : [0, 2] \mapsto \mathbb{R}$

$$f(x) = \begin{cases} 0 & (0 \le x < 1), \\ 1 & (x = 1), \\ 2 & (1 < x \le 2) \end{cases}$$

when $\xi = 1$.
- Thus we introduce a variant of our definition of limit.

# One Sided Limits

- **Definition 7.7. Limit from above and below.** *Suppose that* $\mathcal{A} \subset \mathbb{R}$ *and* $\mathcal{B} \subset \mathbb{R}$, $f : \mathcal{A} \mapsto \mathcal{B}$, $a < \xi$ *and* $(a, \xi) \in \mathcal{A}$. *Then* $\lim_{x \to \xi -} f(x) = \ell$ *means that there is an* $\ell \in \mathbb{R}$ *such that for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *so that whenever* $x \in \mathcal{A}$ *and* $\xi - \delta < x < \xi$ *we have* $|f(x) - \ell| < \varepsilon$ *and we call* $\ell$ *the limit from below.*

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- *There is a corresponding definition for limit from above. Suppose that* $\mathcal{A} \subset \mathbb{R}$ *and* $\mathcal{B} \subset \mathbb{R}$, $f : \mathcal{A} \mapsto \mathcal{B}$, $\xi < b$ *and* $(\xi, b) \in \mathcal{A}$. *Then* $\lim_{x\to\xi+} f(x) = \ell$ *means that there is an* $\ell \in \mathbb{R}$ *such that for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *so that whenever* $x \in \mathcal{A}$ *and* $\xi < x < \xi + \delta$ *we have* $|f(x) - \ell| < \varepsilon$ *and we call* $\ell$ *the limit from above.*

- **Example 7.9.** *Suppose that* $f : [0, \infty) \mapsto \mathbb{R} : f(x) = \sqrt{x}$. *Then* $\lim_{x \to 0+} f(x) = 0$.

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- *Proof.* Let $\varepsilon > 0$. Choose $\delta = \varepsilon^2$. Then, whenever $0 < x < \delta$ we have

$$|f(x) - 0| = \sqrt{x} < \sqrt{\delta} = \varepsilon.$$

Note that $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0-} f(x)$ do not exist.

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- As might be expected, if the limits from below and above exist and agree, then the limit does exist.

**Theorem 3**

*Suppose* $a < \xi < b$ *and* $f : (a, b) \mapsto \mathbb{R}$*. Then* $\lim_{x \to \xi} f(x)$ *exists and converges to* $\ell$ *if and only if both the limits*

$$\lim_{x \to \xi-} f(x), \quad \lim_{x \to \xi+} f(x)$$

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- The proof is immediate on comparing the definitions.