Introduction to Analysis: Series

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- Thus given a sequence $\langle a_n \rangle$ we can form a new sequence $\langle s_n \rangle$ defined by

$$s_n = \sum_{m=1}^{n} a_m. \tag{1.1}$$

- **Definition 6.1** *If the sequence* $\langle s_n \rangle$ *converges, then we say that the* **infinite series**

$$\sum_{m=1}^{\infty} a_m = a_1 + a_2 + \cdots + a_n + \cdots \tag{1.2}$$

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- *The* s_n *are called the* **partial sums** *of the infinite series.*
- *When a series converges the sum*

$$t_n = \sum_{m=n+1}^{\infty} a_n \tag{1.3}$$

is called the **tail** *of the series.*

- **Remark** *There is no reason that a series has to start with* $n = 1$*. We could equally work with*

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- *Moreover if we can establish the convergence for some* M *, then it follows for any* M *by adding or subtracting a finite number of terms.*

- **Example 6.1.** *Let* $x \in \mathbb{R}$ *and* $a_n = x^n$, *so that*

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- *Since a sequence cannot have two limits the series again diverges, even though it is bounded.*
- *Thus we conclude that* $\sum_{n=1}^{\infty} x^n$ *converges if and only if* $|x| < 1$, *and in that case it sums to* $\frac{x}{1-x}$.

- **Example 6.2.** *Let* $a_n = \big(n(n+1)\big)^{-1}$. *Then*

$$s_n = \sum_{m=1}^{n} a_m = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)}$$

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- *Now suppose the formula has been verified for n. Then*

$$s_{n+1} = s_n + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)}$$

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- *Now we let* $n \to \infty$. *Thus* $s_n \to 1$. *Hence*

$$\sum_{m=1}^{\infty} \frac{1}{m(m+1)} = 1.$$

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- *Moreover, when* $m \geq 2$ *we have* $\dfrac{1}{m^2} \leq \dfrac{1}{m(m-1)}$ *so*

$$\begin{aligned} u_n &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &\leq 1 + \frac{1}{1.2} + \frac{1}{2.3} + \cdots \frac{1}{(n-1)n} \\ &= 1 + s_{n-1} \end{aligned}$$

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- *Hence we have an increasing sequence bounded above.*
- *Thus by the monotonic convergence theorem* u_n *converges.*
- *This is yet another example where we have established convergence but do not yet have the tools to give the value of the limit.*

- An immediate consequence of the definition.

Theorem 1

Suppose that

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converges. Then the tail of the series

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- *Proof.* Let ℓ denote the value of the infinite series. Then

$$t_n = \ell - s_n \to 0 \text{ and } a_n = t_{n-1} - t_n.$$

Combination Theorem for Series

- We can now port over the theory of sequences. For example the following is immediate.

Theorem 2 (The Combination Theorem for Series)

Suppose that

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n$$

converge to α *and* β *respectively and* λ *and* μ *are real numbers. Let*

$$c_n = \lambda a_n + \mu b_n \, (n \in \mathbb{N}).$$

Then

$$\sum_{n=1}^{\infty} c_n$$

converges to $\lambda\alpha + \mu\beta$*.*

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- There are four steps to the algorithm. If the algorithm fails to determine the convergence or divergence, then an *ad hoc* method will be required.

Algorithm to Test Series for Convergence

- **Step 1.** *If* $\lim_{n\to} a_n$ *does not exist, or it does but it is not* 0*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

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- **2.2.** Suppose that $0 \le c_n \le a_n$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n$ diverges. Then so does $\sum_{n=1}^{\infty} a_n$.

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- **Step 3.** *The ratio test.* Suppose that $a_n \neq 0$ for every large n and $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists. Let its value be ℓ.
  If $\ell < 1$, then $\sum_{n=1}^{\infty} a_n$ converges and if $\ell > 1$, then it diverges.
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- There are more sophisticated versions of **3.**, e.g. the n-th root test, but if Step **3.** fails these other versions are unlikely to do any better.
- **Step 4.** *The Leibnitz (or alternating series) test.* Suppose there is a sequence $\langle d_n \rangle$ which is (i) non-negative, (ii) decreasing and (iii) satisfies $\lim_{n\to\infty} d_n = 0$ and (iv) $a_n = (-1)^{n-1} d_n$. Then $\sum_{n=1}^{\infty} a_n$ converges.

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- Example 6.3. $\sum_{m=1}^{\infty}\frac{1}{m^2}$ gives an example of convergence by the comparison test.
- Crucial for comparison is a range of useful examples.
- We will show later that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. Then it follows from part 2 of the comparison test that if $c < 1$, then $\sum_{n=1}^{\infty}\frac{1}{n^c}$ diverges.

- **Example 6.6.** *Let* $a_n = (n!)^2/(2n)!$. *Then*

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n)!((n+1)!)^2}{(2n+2)!(n!)^2} = \frac{(n+1)^2}{(2n+1)(2n+2)} \to \frac{1}{4}.$$

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- *Hence* $\sum_{n=1}^{\infty} a_n$ *converges by the ratio test.*
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- **Example 6.7** *Let* $x \in \mathbb{R}$ *and* $b_n = (n!)^2 x^n/(2n)!$. *Then*

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- *So* $\sum_{n=1}^{\infty} b_n$ *converges for* $|x| < 4$ *and diverges for* $|x| > 4$.
- Note that nothing can be concluded when $|x| = \frac{1}{4}$.
- By more sophisticated arguments the series can be shown to converge when $x = -\frac{1}{4}$ and diverge when $x = \frac{1}{4}$.

- **Example 6.8.** *Let* $x \in \mathbb{R}$ *and* $c_n = \dfrac{x^n}{n!}$.

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regardless of the value of x.

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- Note that here we have deployed the conventions $0! = 1$ and that in such series $x^0 = 1$ even when $x = 0$.

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- This explains why the ratio test cannot have any conclusion when the limit is 1.

- **Example 6.10.** Let

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- Thus

$$\sum_{n=1}^{\infty} a_n$$

converges by the Leibnitz test.

Proofs of the Tests

- The first test is easily dealt with.

Theorem 3

If $\lim_{n\to\infty} a_n$ *does not exist, or it does but is not* 0, *then*

$$\sum_{n=1}^{\infty} a_n$$

diverges.

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- The first test is easily dealt with.

Theorem 3

If $\lim_{n\to\infty} a_n$ *does not exist, or it does but is not* 0, *then*

$$\sum_{n=1}^{\infty} a_n$$

diverges.

- *Proof.* Suppose on the contrary that $s_n = \sum_{m=1}^{n} a_m$ converges. Then, by Thoerem 6.1

$$\lim_{n\to\infty} a_n = 0$$

contradicting the hypothesis.

Proofs of the Tests

- The remaining tests are more demanding.

Theorem 4

1. Suppose that $|a_n| \le b_n$ *for every* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} b_n$ *converges. Then so does* $\sum_{n=1}^{\infty} a_n$.

2. Suppose that $0 \le c_n \le a_n$ *for every* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} c_n$ *diverges. Then so does* $\sum_{n=1}^{\infty} a_n$.

Proofs of the Tests

- Restatement of Theorem 6.4.1. Suppose that $|a_n| \le b_n$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges. Then so does $\sum_{n=1}^{\infty} a_n$.

Proofs of the Tests

- Restatement of Theorem 6.4.1. Suppose that $|a_n| \le b_n$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges. Then so does $\sum_{n=1}^{\infty} a_n$.
- *Proof of 1.* We first treat a special case. Suppose $0 \le A_n \le b_n$. Let $u_n = \sum_{m=1}^{n} A_m$ and $B = \sum_{m=1}^{\infty} b_m$. Then

$$u_n \le \sum_{m=1}^{n} b_m \le \sum_{m=1}^{\infty} b_n = B,$$

so $\langle u_n \rangle$ is bounded above.

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- As the A_n are non-negative, the sequence is increasing.

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  so $\langle u_n \rangle$ is bounded above.
- As the A_n are non-negative, the sequence is increasing.
- Hence $\langle u_n \rangle$ converges.
- Now we turn to the general case $|a_n| \le b_n$ for every $n \in \mathbb{N}$.

Proofs of the Tests

- Let

$$D_n = \begin{cases} a_n & \text{if } (a_n \geq 0), \\ 0 & \text{if } (a_n < 0), \end{cases} \quad E_n = \begin{cases} 0 & \text{if } (a_n \geq 0), \\ -a_n & \text{if } (a_n < 0). \end{cases}$$

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- Then $0 \leq D_n \leq b_n$ and $0 \leq E_n \leq b_n$.
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$$\sum_{n=1}^{\infty} D_n \text{ and } \sum_{n=1}^{\infty} E_n$$

both converge.

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- Thus by the combination theorem, Theorem 2,

$$\sum_{n=1}^{\infty} (D_n - E_n)$$

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both converge.
- Thus by the combination theorem, Theorem 2,

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converges.
- But $D_n - E_n = a_n$ for every $n \in \mathbb{N}$.

Proofs of the Tests

- Restatement of Theorem 6.4.2. Suppose that $0 \le c_n \le a_n$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n$ diverges. Then so does $\sum_{n=1}^{\infty} a_n$.

Proofs of the Tests

- Restatement of Theorem 6.4.2. Suppose that $0 \le c_n \le a_n$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n$ diverges. Then so does $\sum_{n=1}^{\infty} a_n$.
- *Proof of 2.* Let

$$t_n = \sum_{m=1}^{n} c_m.$$

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- *Proof of 2.* Let
$$t_n = \sum_{m=1}^{n} c_m.$$
- Since each $c_m \ge 0$, t_n is an increasing sequence.
- If the sequence $\langle t_n \rangle$ were bounded then the series $\sum_{n=1}^{\infty} c_n$ would have to converge.
- Hence it is unbounded.

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- Since each $c_m \ge 0$, t_n is an increasing sequence.
- If the sequence $\langle t_n \rangle$ were bounded then the series $\sum_{n=1}^{\infty} c_n$ would have to converge.
- Hence it is unbounded.
- But $s_n \ge t_n$, so $\langle s_n \rangle$ is unbounded and hence $\sum_{n=1}^{\infty} a_n$ diverges.

Proofs of the Tests

- The Ratio Test.

Theorem 5

Suppose that $a_n \neq 0$ for every large n and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists. Let its value be ℓ.

If $\ell < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\ell > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof of the Ratio Test

- We assume that $\lim_{n\to\infty} |a_{n+1}/a_n| = \ell$. Of course $\ell \geq 0$.

Proof of the Ratio Test

- We assume that $\lim_{n\to\infty} |a_{n+1}/a_n| = \ell$. Of course $\ell \geq 0$.
- Suppose first $\ell < 1$. The plan is to compare with $\sum_{n=1}^{\infty} x^n$.

Proof of the Ratio Test

- We assume that $\lim_{n\to\infty} |a_{n+1}/a_n| = \ell$. Of course $\ell \geq 0$.
- Suppose first $\ell < 1$. The plan is to compare with $\sum_{n=1}^{\infty} x^n$.
- Let $\varepsilon = \frac{1-\ell}{2}$ and choose $N \in \mathbb{N}$ so that whenever $n > N$ we have $||a_{n+1}/a_n| - \ell| < \varepsilon$ and so $|a_{n+1}/a_n| - \ell < \varepsilon$.

Proof of the Ratio Test

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- Put $x = \ell + \varepsilon$ so that $x = \ell + \dfrac{1-\ell}{2} = \dfrac{1+\ell}{2} < 1$ and $|a_{n+1}/a_n| < x$ whenever $n > N$.

Proof of the Ratio Test

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- Put $x = \ell + \varepsilon$ so that $x = \ell + \dfrac{1-\ell}{2} = \dfrac{1+\ell}{2} < 1$ and $|a_{n+1}/a_n| < x$ whenever $n > N$.
- Now by induction on $n \geq N$ we have $|a_n| \leq x^n |a_N| x^{-N}$.

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- Put $x = \ell + \varepsilon$ so that $x = \ell + \dfrac{1-\ell}{2} = \dfrac{1+\ell}{2} < 1$ and $|a_{n+1}/a_n| < x$ whenever $n > N$.
- Now by induction on $n \geq N$ we have $|a_n| \leq x^n |a_N| x^{-N}$.
- To see this take the base case as $n = N$ and then given $n \leq N$ we have $|a_{n+1}| < x|a_n| \leq x^{n+1}|a_N|x^{-N}$.

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- Let $\varepsilon = \frac{1-\ell}{2}$ and choose $N \in \mathbb{N}$ so that whenever $n > N$ we have $\big||a_{n+1}/a_n| - \ell\big| < \varepsilon$ and so $|a_{n+1}/a_n| - \ell < \varepsilon$.
- Put $x = \ell + \varepsilon$ so that $x = \ell + \dfrac{1-\ell}{2} = \dfrac{1+\ell}{2} < 1$ and $|a_{n+1}/a_n| < x$ whenever $n > N$.
- Now by induction on $n \ge N$ we have $|a_n| \le x^n |a_N| x^{-N}$.
- To see this take the base case as $n = N$ and then given $n \le N$ we have $|a_{n+1}| < x|a_n| \le x^{n+1}|a_N|x^{-N}$.
- By Example 6.1 $\sum_{n=1}^{\infty} x^n$ converges. Hence $\sum_{n=1}^{\infty} x^n |a_N| x^{-N}$ converges. Thus, by comparison $\sum_{n=N}^{\infty} a_n$ converges.

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- Then, by taking $\varepsilon = \ell - 1$ in the definition of convergence it follows that there is an $N \in \mathbb{N}$ so that whenever $n \geq N$ we have
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Proof of the Ratio Test

- We assume that $\lim_{n\to\infty} |a_{n+1}/a_n| = \ell$.
- Now suppose that $\ell > 1$.
- Then, by taking $\varepsilon = \ell - 1$ in the definition of convergence it follows that there is an $N \in \mathbb{N}$ so that whenever $n \geq N$ we have
$$\left|\frac{a_{n+1}}{a_n}\right| > 1.$$
- Hence
$$|a_{n+1}| > |a_n| > \ldots |a_N| > 0.$$
- Thus either $\lim_{n\to\infty} a_n$ does not exist or $|\lim_{n\to\infty} a_n| \geq |a_N| > 0$,
- so the second part of the theorem follows from Theorem 6.2.

The Root Test

- The following test is not part of the algorithm. For most applications it is easier to use the ratio test. It does have the merit of not requiring $a_n \neq 0$ and there is an important application later to power series.

Theorem 6

If the sequence $b_n = |a_n|^{1/n}$ *is bounded and* $\limsup_{n\to\infty} b_n < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely. If* $\langle b_n \rangle$ *is unbounded, or it is bounded but* $\limsup_{n\to\infty} b_n > 1$, *then the series diverges.*

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- I will skip the proof. It can be read in the course text.

- We now come to the final part of our algorithm.

Theorem 7 (The Leibnitz Test)

Suppose $\langle d_n \rangle$ is (i) non-negative, (ii) decreasing, (iii) satisfies $\lim_{n\to\infty} d_n = 0$, and (iv) $a_n = (-1)^{n-1} d_n$. Then $\sum_{n=1}^{\infty} a_n$ converges.

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- We also have $s_{2n} = s_{2n-1} + a_{2n} = s_{2n-1} - d_{2n} \le s_{2n-1}$ so that
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$s_2 \le s_4 \le s_6 \le \ldots \le s_{2n} \le s_{2n-1} \le \cdots \le s_5 \le s_3 \le s_1$.
- Thus $\langle s_{2n}\rangle$ is bounded above by s_1 and $\langle s_{2n-1}\rangle$ is bounded below by s_2, and so both converge, to, say, ℓ_1 and ℓ_2. Then $\ell_1 - \ell_2 = \lim_{n\to\infty}(s_{2n-1} - s_{2n}) = \lim_{n\to\infty} d_{2n} = 0$. Let $\ell = \ell_1 = \ell_2$. Then $\lim_{n\to\infty} s_n = \ell$.

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- but
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$
diverges since the n-th partial sum is bounded below by $\sqrt{n}$ and so is unbounded.

- The following is a corollary of the comparison test.

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Theorem 9

Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$. That is, f is a bijection - for every $n \in \mathbb{N}$ there is a unique $m \in \mathbb{N}$ such that $f(m) = n$. Suppose, moreover, that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely. Then so does $\displaystyle\sum_{n=1}^{\infty} a_{f(n)}$ and $\displaystyle\sum_{n=1}^{\infty} a_{f(n)} = \sum_{n=1}^{\infty} a_n$.

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- However one rearranges an absolutely convergent series the sum is the same. This is false for conditional convergence.

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- Given ε and a suitable N one needs to choose an M so that $f(m) > N$ when $m > M$.

- Often the ratio test is useless, because the comparison is with a series which converge or diverges exponentially fast. Most series converge or diverge much more slowly. The series considered below are more useful.

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Suppose that $\sigma \in \mathbb{R}$ and $\sigma \le 1$. Then the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\sigma}}$ diverges.

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- Therefore $\lim_{n\to\infty} (s_{2n} - s_n) = \ell - \ell = 0$. But

$$s_{2n} - s_n = \sum_{m=n+1}^{2n} \frac{1}{m^\sigma} \ge \sum_{m=n+1}^{2n} \frac{1}{(2n)^\sigma} = 2^{-\sigma} n^{1-\sigma} \ge \frac{1}{2}.$$

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- Taking limits we just showed that $0 \ge \frac{1}{2}$.

- One can contrast the previous theorem with the next one.

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- *Proof.* We have $n^\sigma > 0$ for every $n \in \mathbb{N}$. Thus the partial sums $s_n = \sum_{m=1}^{n} \frac{1}{m^\sigma}$ form an increasing sequence.
- Hence it suffices to show that the subsequence $\langle s_{2^k} \rangle$ is bounded above, i.e. $s_{2_k} \le B$ for every $k \in \mathbb{N}$, because given n the Archimedean property ensures that there is a k with $n \le 2^k$ and then it follows that $s_n \le s_{2^k} \le B$.

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- Then $1 + t_1 + t_2 + \cdots t_k$

$$= 1+(s_2-s_1)+\cdots(s_{2^k}-s_{2^{k-1}}) = s_{2^k}+1-s_1 = s_{2^k}. \quad (4.6)$$

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- By Example 6.1 and the comparison test, $\sum_{j=1}^{k} t_j$ converges and so by (4.6) $\langle s_{2^k} \rangle$ converges and so is bounded, as required.

- We now examine a special class of series which give rise to many of the most important functions in mathematics and have myriad applications.

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- **Definition6.3.** *For a given sequence $\langle a_n \rangle$ of real numbers consider the series*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{5.7}$$

- *We call such a series a* **power series**. *Note that we include a term with $n = 0$ and by convention $x^0 = 1$ regardless of the value of x.*

- The following is the fundamental theorem of power series.

Theorem 12

Given a sequence $\langle a_n \rangle$ of real numbers and the corresponding power series $A(x)$,
(i) the series converges absolutely for every x and

$$\limsup_{n\to\infty} |a_n|^{1/n} = 0$$

or (ii) there is a positive real number R such that the series converges absolutely for all x with $|x| < R$ and diverges for all x with $|x| > R$ and $\limsup_{n\to\infty} |a_n|^{1/n} = R^{-1}$
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- **Definition 6.4.** *It is conventional to define R in case (ii) to be the* **radius of convergence** *of $A(x)$, and to extend this to be $R = \infty$ in case (i) and $R = 0$ in case (iii). By an abuse of notation we could write $R = 1/\limsup_{n\to\infty} |a_n|^{1/n}$.*

Proof of Theorem 6.12

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- Then $\limsup_{n\to\infty} |c_n|^{1/n} = |x|R^{-1}$ and by the root test the series converges absolutely when $|x| < R$ and diverges when $|x| > R$. which gives (ii).
- Finally, if $\limsup_{n\to\infty} |c_n|^{1/n} = 0$, then $|x| \limsup_{n\to\infty} |a_n|^{1/n} = 0$ and so $\limsup_{n\to\infty} |a_n|^{1/n} = 0$.

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- If $\langle |c_n|^{1/n} \rangle$ is unbounded, then so is $\langle |a_n|^{1/n} \rangle$ and by the root test the series diverges for all $x \neq 0$, which gives case (iii).
- If $\limsup\limits_{n\to\infty} |c_n|^{1/n}$, exists and is non-zero, then likewise for $\limsup\limits_{n\to\infty} |a_n|^{1/n}$ and we can define $R = \big(\limsup\limits_{n\to\infty} |a_n|^{1/n}\big)^{-1}$.
- Then $\limsup\limits_{n\to\infty} |c_n|^{1/n} = |x|R^{-1}$ and by the root test the series converges absolutely when $|x| < R$ and diverges when $|x| > R$. which gives (ii).
- Finally, if $\limsup\limits_{n\to\infty} |c_n|^{1/n} = 0$, then $|x| \limsup\limits_{n\to\infty} |a_n|^{1/n} = 0$ and so $\limsup\limits_{n\to\infty} |a_n|^{1/n} = 0$.
- Thus by the root test the series converges absolutely for every x, which gives case (i) and completes the proof.

Important Functions

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- **Definition 6.5.** *Whenever the corresponding series converges we define*

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

- The first part of the following theorem is an easy consequence of the ratio test and the second part is obvious.

Theorem 13

(i) Each of the series defining exp, sin *and* cos *has radius of convergence* ∞.
(ii) We have $\exp(0) = 1$, $\sin(0) = 0$, $\cos(0) = 1$.
(iii) For every pair of real numbers x *and* y *we have*

$$\exp(x + y) = \exp(x)\exp(y)$$

and

$$\exp(-x) = \frac{1}{\exp(x)}.$$

(iv) For every $x \in \mathbb{R}$ *we have* $\exp(x) > 0$.
(v) The function $\exp(x)$ *is unbounded above, and for every* $\varepsilon > 0$ *there are* x *such that* $\exp(x) < \varepsilon$.

- It remains to prove the following.
  (iii) For every pair of real numbers x and y we have

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  (v) The function $\exp(x)$ is unbounded above, and for every $\varepsilon > 0$ there are x such that $\exp(x) < \varepsilon$.
- For the time being assume (iii). When $x \geq 0$ all the terms in the series are non-negative and the first term is 1. Thus in this case $\exp(x) > 0$. By the second equation in (iii) this then follows when $x < 0$, which establishes (iv).

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- For the time being assume (iii). When $x \geq 0$ all the terms in the series are non-negative and the first term is 1. Thus in this case $\exp(x) > 0$. By the second equation in (iii) this then follows when $x < 0$, which establishes (iv).
- For any $n \in \mathbb{N}$ we have $\exp(n) = 1 + n + \cdots > n$ Hence by the Archimedean property exp is unbounded above. Moreover by the second equation in (iii) we have $\exp(-n) < 1/n$. This establishes (v).

- We now prove (iii), that for every pair x and y we have

$$\exp(x+y) = \exp(x)\exp(y), \quad \exp(-x) = 1/\exp(x).$$

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$$\exp(x+y) = \exp(x)\exp(y), \quad \exp(-x) = 1/\exp(x).$$
- By the ratio test $\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{|x|^m|y|^k}{m!k!}$ converge absolutely and so by the rearrangement theorem
$$\exp(x)\exp(y) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{x^m y^k}{m!k!}$$
can be rearranged in any way we like.

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can be rearranged in any way we like.

- Thus it is $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\substack{k=0\\m+k=n}}^{\infty}\frac{x^my^k}{m!k!}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{x^my^{n-m}}{m!(n-m)!}$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{m=0}^{n}\frac{n!x^my^{n-m}}{m!(n-m)!}=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}x^my^{n-m}$$

$$=\sum_{n=0}^{\infty}\frac{(x+y)^n}{n!}=\exp(x+y).$$

- There is one other interesting theorem in this chapter.

Theorem 14

Suppose that $x \in \mathbb{R}$. Then

$$\lim_{n\to\infty}(1+x/n)^n = \lim_{n\to\infty}(1-x/n)^{-n} = \exp(x).$$

- There is one other interesting theorem in this chapter.

Theorem 14

Suppose that $x \in \mathbb{R}$. Then

$$\lim_{n\to\infty}(1+x/n)^n = \lim_{n\to\infty}(1-x/n)^{-n} = \exp(x).$$

- I will not prove it here, but the details can be found in the course text.