Introduction to Analysis: Monotonic Sequences, and Subsequences

Robert C. Vaughan

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Monotonic Sequences

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- *2. When* $a_n < a_{n+1}$ *for every* $n \in \mathbb{N}$ *we call it* **strictly increasing***, and on the other hand when* $a_n > a_{n+1}$ *for every* $n \in \mathbb{N}$ *we call it* **strictly decreasing**.

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- *3. Such sequences are called* **monotonic** *in case 1. and* **strictly monotonic** *in case 2.*
- There is a modern tendency to use increasing to mean strictly increasing and, by a terrible misuse of language, to use non-decreasing to mean increasing, and a concomitant variant for the other two cases. A student of the English language would expect that the sequence $\langle (-1)^n \rangle$ is non-decreasing and non-increasing.

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- *All the above, except 4. are monotonic, 2. and 3. are strictly monotonic*

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Theorem 1

Every monotonic bounded sequence $\langle a_n \rangle$ *converges. When it is increasing the limit is given by* $\sup\{a_n : n \in \mathbb{N}\}$ *and when it is decreasing it is given by* $\inf\{a_n : n \in \mathbb{N}\}$*.*

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- Hence there exists an $N \in \mathbb{N}$ so that $A - \varepsilon < a_N \le A$.
- As $\langle a_n \rangle$ is increasing, induction gives $A - \varepsilon < a_{N+n} \le A$ for every $n \in \mathbb{N}$, so $|a_n - A| < \varepsilon$ for every $n > N$.

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- Solving for ℓ, $\frac{1}{2}\ell = \frac{1}{\ell}, \ell^2 = 2$. So $\sqrt{2}$ exists and $= \ell$.

Subsequences

- **Definition 5.2.** *Suppose that* $\langle a_n \rangle$ *is a sequence and* $\langle m_n \rangle$ *is a strictly increasing sequence of natural numbers. That is,* $m_n \in \mathbb{N}$ *and* $m_n < m_{n+1}$ *for every* $n \in \mathbb{N}$*. Then we call the sequence* $\langle a_{m_n} \rangle$ *a subsequence of* $\langle a_n \rangle$*.*

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and $m_n = n^2$, so that $\langle m_n \rangle = 1, 4, 9, 16, \ldots$. Then

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- Subsequences are very useful as a "way in" to the behaviour of a sequence, since a nasty subsequence may well have subsequences which are much easier to deal with and then give us a handle on the original sequence.

- We can make good use of the following theorem.

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- Let $\varepsilon > 0$ and choose N so that whenever $n > N$ we have $|a_n - \ell| < \varepsilon$.
- Since $m_n \geq n$ we also have $m_n > N$ when $n > N$.
- Therefore for every $n > N$ we have $|a_{m_n} - \ell| < \varepsilon$ and so $\langle a_{m_n} \rangle$ converges to ℓ.

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- *Proof.* Suppose on the contrary that the sequence converges to ℓ.
- Then the subsequences $\langle(-1)^{2n}\rangle$ and $\langle(-1)^{2n-1}\rangle$ would both converge to ℓ.
- But the first one converges to $+1$ and the second one to -1 and this would contradict Theorem 4.2.

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- Thus a_n converges and as $a_n > 1$ we have $\lim_{n\to\infty} a_n \geq 1$.

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Every bounded sequence has a convergent subsequence.

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- Thus $\langle a_{m_k} \rangle$ is a decreasing sequence.
- Now suppose there are at most a finite number of extrema.

- Repeat.
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- Thus in this case we have constructed an increasing sequence.

- **Example 5.6** *1. Recall the Example* **4.2**, *which we examined in detail in Example* **5.5** *where we defined inductively*

$$x_1 = 2, x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

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- *Hence, by induction,* x_n *is bounded between* 1 *and* 2*. Thus the sequence has a convergent subsequence.*
- *2. In the example* $\langle(-1)^n\rangle$ *we looked at in Examples 4.7 and 8, each of the subsequences* $\langle(-1)^{2n}\rangle$ *and* $\langle(-1)^{2n-1}\rangle$ *are monotonic and convergent.*

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- If the sequence is also bounded below, then each of the sets $\mathcal{A}_n$ is bounded below by the same bound.
- Hence $\langle t_n \rangle$ is decreasing and bounded below and so convergent.
- A similar argument shows that s_n is increasing and bounded above, and so convergent.

- Definition 5.3. *When a sequence $\langle a_n \rangle$ is bounded we define*

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} t_n = \lim_{n\to\infty} \sup\{a_m : m \ge n\}$$

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- The important thing is that when a sequence is bounded these limits always exist.
- Moreover if we were to adopt a general version of the convention mentioned above, then we could say that they exist even when the sequence is unbounded. This can be very useful and avoids having to deal with objects which might not exist.

- **Example 5.7.** *Let $\langle a_n \rangle$ be bounded and let $\langle a_{m_n} \rangle$ be a convergent subsequence. Then*

$$\liminf_{n\to\infty} a_n \le \lim_{n\to\infty} a_{m_n} \le \limsup_{n\to\infty} a_n.$$

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- and the conclusion follows by Corollary 4.7.

- The power of the concept is illustrated by Theorem 5.

Theorem 5

Suppose that $\langle a_n \rangle$ is bounded. Then it converges if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ and then it converges to the common value.

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Suppose that $\langle a_n \rangle$ is bounded. Then it converges if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ and then it converges to the common value.

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- Since these bounds hold for every element of $\mathcal{A}_n$, in the notation used in the preamble we have

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- Another $m_k = \frac{k(k-1)}{2} + k = \frac{(k+1)k}{2}$, $a_{m_k} = \frac{k}{k} = 1$ and this converges to 1.
- It follows that $\liminf_{n\to\infty} a_n = 0$, $\limsup_{n\to\infty} a_n = 1$.

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- We remark that in order to satisfy the criterion for being a Cauchy sequence it suffices to know that the above holds just for $n > m > N$ because that gives the case $m < n$, the case $n < m$ holds by interchanging the values of m and n, and the case $m = n$ is clear.

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- Then for any m, n with $n > N$, $m > N$, by the triangle inequality, $|a_n - a_m| =$

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- *We have, for* $n > m \geq 1$, *by generalizing the triangle inequality to* $n - m$ *terms (an easy induction),* $|a_n - a_m|$

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- For example, take $a_n = \ell + \lambda^n$.