Introduction to Analysis: Sequences

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February 23, 2024

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- The exception is 2. where we have $\mathcal{A} = \{1\}$.

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- Recalling Definitions 2.6, 2.7, 2.8 we have at once the following theorem.

Theorem 1

A sequence $\langle a_n \rangle$ is bounded if and only if there is a real number H such that for every $n \in \mathbb{N}$ we have $|a_n| \le H$.

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- This leads on naturally to the next topic

Convergent Sequences

- A sequence $\langle a_n \rangle$ converges to the limit ℓ (where $\ell \in \mathbb{R}$) when the following holds.
  Definition 4.2 *Given any real number $\varepsilon > 0$ there is a real number N such that whenever $n \in \mathbb{N}$ and $n > N$ we have*

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- Note that in general we would expect that N is a function of ε.
- Occasionally we can only prove its existence, but those proofs are usually pretty tricky.

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- Often in order to make progress one will need to have a good guess for ℓ.
- Later we will see ways which avoid this.

- **Example 4.3.** *1. Let* $a_n = 1/n$*. We would guess that the limit exists and is* 0*.*
  2. Suppose that b_n *is a constant sequence, i.e there is a real number* c *such that for every* $n \in \mathbb{N}$ *we have* $b_n = c$*. Then* $\lim_{n\to\infty} b_n = c$*.*

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- 2. is even easier. Let $\varepsilon > 0$ and choose $N = 1$, say. Then, whenever $n > N$ we have

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and we are done.

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- Generally shifting the suffix by a constant amount does not change the convergence.

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- We are going to see many appearances by the triangle inequality in convergence proofs.

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- Note that it diverges even though it is bounded. In other words being bounded is not enough to confer convergence on a sequence.

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Theorem 4 (The Combination Theorem for sequences)

Suppose that $\langle a_n \rangle$ converges to α and $\langle b_n \rangle$ converges to β as $n \to \infty$, and let λ and μ be real numbers. Then
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- In part (iii) there is a convention. Since $\beta \neq 0$ we are confident that there is some N_0 so that for $n > N_0$ we have $b_n \neq 0$. It is possible there are $n \leq N_0$ with $b_n = 0$. In that case the convention is that we suppose that $n > N_0$ and ignore the $n \leq N_0$.

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  (iii) If $\beta\neq 0$, then $\frac{a_n}{b_n}\to\frac{\alpha}{\beta}$ as $n\to\infty$.
- *Proof.* (iii) In view of (ii), it suffices to prove that

$$\frac{1}{b_n}\to\frac{1}{\beta} \text{ as } n\to\infty.$$

- Somehow we need to make use of $\beta\neq 0$. From the special case $\varepsilon=\frac12|\beta|$ we know that there is an N_1 such that whenever $n>N_1$ we have $|b_n-\beta|<\frac12|\beta|$ so that by the triangle inequality we have $|b_n|>|\beta|/2$.
- Now choose an arbitrary $\varepsilon>0$ and N_2 so that whenever $n>N_2$ we have $|b_n-\beta|<\varepsilon|\beta|^2/(2)$.
- Let $N=\max\{N_1,N_2\}$.
- Then whenever $n>N$ we have

$$\left|\frac{1}{b_n}-\frac{1}{\beta}\right|=\left|\frac{\beta-b_n}{b_n\beta}\right|=\frac{|\beta-b_n|}{|b_n||\beta|}<\frac{2|\beta-b_n|}{|\beta|^2}<\varepsilon.$$

- **Example 4.8.** Prove that

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- and we know from Example **4.3.** 1. that $n^{-1} \to 0$ as $n \to \infty$ and that $\lim_{n\to\infty} c = c$.
- Hence we can apply Theorem **4.4** multiple times and obtain successively

$$n^{-2} \to 0, \quad n^{-3} \to 0, \quad n^{-4} \to 0,$$
$$1 - 3n^{-2} + 5n^{-4} \to 1,$$
$$4 + 5n^{-1} - 3n^{-3} \to 4,$$
$$\frac{1 - 3n^{-2} + 5n^{-4}}{4 + 5n^{-1} - 3n^{-3}} \to \frac{1}{4}.$$

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Suppose that $\langle a_n \rangle$, $\langle b_n \rangle$, $\langle c_n \rangle$ are three real sequences with $a_n \le b_n \le c_n$ for every $n \in \mathbb{N}$, and $a_n \to \ell$ as $n \to \infty$ and $c_n \to \ell$ as $n \to \infty$. Then $b_n \to \ell$ as $n \to \infty$

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- Let $\varepsilon > 0$. Choose N_1 so that whenever $n > N_1$ we have $|a_n - \ell| < \varepsilon$ and choose N_2 so that whenever $n > N_2$ we have $|c_n - \ell| < \varepsilon$.

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- Then, whenever $n > N$, we have

$$-\varepsilon < a_n - \ell \le b_n - \ell \le c_n - \ell < \varepsilon,$$
$$-\varepsilon < b_n - \ell < \varepsilon,$$
$$|b_n - \ell| < \varepsilon.$$

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- Now both sides have limit 0 so we can apply the sandwich theorem.

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- Hence, by the combination theorem we have the desired conclusion.

Divergence to Infinity

- **Definition 4.3.** *A sequence* $\langle a_n \rangle$ **diverges** *to* $+\infty$ *(written* $x_n \to +\infty$*) as* $n \to \infty$ *when for any* $B > 0$ *there exists a real number* N *such that whenever* $n > N$ *we have* $a_n > B$*. Likewise* $\langle a_n \rangle$ **diverges** *to* $-\infty$ *(and we write* $x_n \to -\infty$*) as* $n \to \infty$ *when for any* $b < 0$ *there exists a real number* N *such that whenever* $n > N$ *we have* $a_n < b$*.*

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- **Example 4.11** *1. Let* $a_n = \sqrt{n}$ *for* $n \in \mathbb{N}$*. Then* $\langle a_n \rangle$ *diverges to* $+\infty$*.*
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- *Proof.* 1. Let $B > 0$ and choose $N = B^2$. Then, whenever $n > N$ we have $a_n = \sqrt{n} > \sqrt{N} = B$.
- 2. Let $B > 0$ and choose $N = (\sqrt{B} + 1)^2$. Then

$$b_n = n + (-1)^n\sqrt{n} \geq n - \sqrt{n} = \left(\sqrt{n} - \tfrac{1}{2}\right)^2 - \tfrac{1}{4}$$
$$> \left(\sqrt{N} - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} = \left(\sqrt{B} + \tfrac{1}{2}\right)^2 - \tfrac{1}{4} = B + \sqrt{B} > B.$$