Introduction to Analysis

The Natural Numbers

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- It is immediate from the fact that $0 < 1$ and the principle of induction that $\mathbb{N}$ is bounded below by 1.

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- **Example 3.1** Let

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- Hence $\mathcal{A}$ is bounded above by 1 and as $1 \in \mathcal{A}$ there cannot be any smaller upper bound.
- Thus $\sup \mathcal{A} = 1$.
- We also have $n \geq 1 > 0$. Thus $1/n > 0$ also, so 0 is a lower bound for $\mathcal{A}$. Hence $\inf \mathcal{A}$ exists. We now show that there is no larger lower bound. We argue by contradiction. Let $b = \inf \mathcal{A}$ and suppose that $b > 0$. Then for every $n \in \mathbb{N}$ we have $b \leq 1/n$. Hence $n \leq 1/b$ which contradicts the Archimedean property.

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- We have not yet defined what we mean by a limit, but it suggests that such a concept is already built in to the definition of the continuum property.

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$$\mathcal{B} = \left\{ \frac{2n}{3n-1} : n \in \mathbb{N} \right\}.$$

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- *Proof.* There are quite a lot of details which need to be attended to.
- We first deal with the upper bound.
- Such proofs should be divided into three parts. (i) Prove that $\mathcal{B} \neq \emptyset$, (ii) prove that 1 is an upper bound, and (iii) prove that there is no smaller upper bound.

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- so that
$$\frac{2n}{3n-1} < b$$
contradicting the assumption on b.

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- 4. 2. shows that, even when a set has an inf. or a sup., that does not guarantee that it has a corresponding min. or max. In other words, extrema may not be members of the set.

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- If $m \geq n$ for every element m of $\mathcal{A}$, then n is a lower bound of $\mathcal{A}$ *and* $n \in \mathcal{A}$, and we would be done.
- If there would be an element m of $\mathcal{A}$ with $m < n$, then we would have $m + 1 \leq n < b + 1$, so that m would satisfy $m < b$ which contradicts b being a lower bound of $\mathcal{A}$.

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- Thus $m \in \mathcal{T}$, so that and $m \leq s + n$ for every $s \in \mathcal{S}$ and $m = s_0 + n$ for some $s_0 \in \mathcal{S}$.

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- Thus $m \in \mathcal{T}$, so that and $m \leq s + n$ for every $s \in \mathcal{S}$ and $m = s_0 + n$ for some $s_0 \in \mathcal{S}$.
- Therefore $m - n \leq s$ for every $s \in \mathcal{S}$ and $m - n = s_0 \in \mathcal{S}$.

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- Let $\mathcal{T} = \{n + s : s \in \mathcal{S}\}$.
- For each $s \in \mathcal{S}$ we have $s + n > b + (-b) = 0$, so that $s + n \geq 1$.
- Hence $\mathcal{T}$ is a subset of $\mathbb{N}$ and so by Theorem 2 has a minimum m.
- Thus $m \in \mathcal{T}$, so that and $m \leq s + n$ for every $s \in \mathcal{S}$ and $m = s_0 + n$ for some $s_0 \in \mathcal{S}$.
- Therefore $m - n \leq s$ for every $s \in \mathcal{S}$ and $m - n = s_0 \in \mathcal{S}$.
- Hence $m - n$ is the minimum for $\mathcal{S}$.

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- so $m - 1 \in \mathcal{A}$ contradicting the minimality of m.
- Thus $na < m < nb$ and dividing by n gives the result.

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- Thus if we can show that
  (i) $P(1)$ is true,
  (ii) whenever $P(n)$ is true $P(n+1)$ is also true,
  then it follows that $\mathcal{S} = \mathbb{N}$ and $P(n)$ is true for every $n \in \mathbb{N}$.

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- Now suppose that $n \ge 4$ and $P(n)$ is true. Then $2^{n+1} =$

$$2.2^n \ge 2n^2 = n^2 + n^2 \ge n^2 + 4n \ge n^2 + 2n + 1 = (n+1)^2.$$

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