Introduction to Analysis

The Real Numbers

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January 24, 2024

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- Since there are quite a number we will divide them into two groups, the Arithmetic axioms and the Order axioms.
- Later we will have to decide what distinguishes $\mathbb{R}$ from $\mathbb{Q}$ and what extra axioms might be required.

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- **Multiplicative inverse.** Given $a \neq 0$ there is an $a^{-1} \in \mathcal{F}$ such that $aa^{-1} = a^{-1}a = 1.$

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- **Example 2.1.** If $x + y = x + z$, then $y = z$.
- *Proof.* We have

$$\begin{aligned}
y &= 0 + y && \text{identity}\\
&= ((-x) + x) + y && \text{inverse}\\
&= (-x) + (x + y) && \text{associative}\\
&= (-x) + (x + z) && \text{hypothesis}\\
&= ((-x) + x) + z && \text{associative}\\
&= 0 + z && \text{inverse}\\
&= z && \text{identity.}
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- The conclusion then follows from the previous example.

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&= (-x)^2 + x((-x) + x) && \text{inverse} \\
&= (-x)^2 + (x(-x) + x^2) && \text{distributive} \\
&= ((-x)^2 + x(-x)) + x^2 && \text{associative} \\
&= ((-x) + x)(-x) + x^2 && \text{distributive} \\
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- Henceforward, apart perhaps from the odd exercise or exam question we will assume that any arithmetical operation we are used to is allowed.

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The symbol $a \leq b$ means $a < b$ or $a = b$.
- The symbol $a > b$ means $b < a$.
- The symbol $a \geq b$ means $b \leq a$.

- By **O1** every element a of $\mathcal{F}$ satisfies exactly one of

$$a < 0, a = 0, 0 < a.$$

The elements with $0 < a$ are called the positive numbers, and those with $a < 0$ are the negative numbers. These two sets, together with the set

$$\{0\}$$

partition $\mathcal{F}$ into three disjoint sets.

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- 2. If $x < 0$, then by Example 2.4, $0 < -x$ and so by part 1. we have

$$0 < (-x)^2 = x^2.$$

Multiplication by negatives

- We have not said anything about multiplication of inequalities by negative numbers. There is good reason for this because the analogue of **O4**
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- This is one of the most common sources of mistakes in mathematics.
- However, we do not need a new axiom. We can deduce the correct conclusion from the axioms we already have.

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- Now we add $ac + bc$ to both sides. Thus, by **O3**,

$$\begin{aligned} bc = bc + 0 = bc + (ac + (-ac)) \\ &= (bc + ac) + (-ac) \\ &< (bc + ac) + (-bc) \\ &= (ac + bc) + (-bc) \\ &= ac + (bc + (-bc)) = ac + 0 = ac \end{aligned}$$

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- The second possibility is $y < x$. Then by the first part of the theorem we would have $y^2 < x^2$ which again contradicts the hypothesis.

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$$\begin{aligned}
(a,b) &= \{x : a < x < b\} && \text{an open interval,}\\
[a,b] &= \{x : a \le x \le b\} && \text{a closed interval,}\\
[a,b) &= \{x : a \le x < b\} && \text{half closed-open interval,}\\
(a,b] &= \{x : a < x \le b\} && \text{half open-closed interval,}\\
(a,\infty) &= \{x : a < x\},\\
[a,\infty) &= \{x : a \le x\},\\
(-\infty,b) &= \{x : x < b\},\\
(-\infty,b] &= \{x : x \le b\}.
\end{aligned}$$

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- *Proof.* By the remark following Example 2.5 we have

$$0 \le (x-y)^2 = x^2 - 2xy + y^2.$$

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- Inequalities are fundamental to analysis and it is desirable to obtain some facility in their manipulation. They can be treated like equations except for the important caveat that multiplication by a negative number can flip an inequality.
- The following is very famous and frequently made use of.

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Suppose that x and y are elements of an ordered field. Then

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- *Proof.* By the remark following Example 2.5 we have
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- Strictly this should be divided into two cases, $<$ and $=$, but with greater familiarity there is less need for pedantry.

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- The following is closely related albeit more complicated.

Theorem 4 (Cauchy-Schwarz)

Suppose that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are $2n$ elements of an ordered field. Then

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- One reason this is important is because it tells us that in n-dimensional Euclidean space the scalar product of two vectors is bounded by the product of their sizes

Cauchy-Schwarz

- *Proof.* Let

$$
\begin{aligned}
A &= a_1^2 + \cdots + a_n^2, \\
B &= a_1 b_1 + \cdots + a_n b_n, \\
C &= b_1^2 + \cdots + b_n^2.
\end{aligned}
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- If $A = 0$, then we have $a_1 = \cdots = a_n = 0$, since otherwise at least one of the terms in A is positive and the others are non-negative and by repeated use of the order axioms A would have to be positive. Thus if $A = 0$, then $B = 0$ and at once

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- Hence we may suppose that $A > 0$.

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To prove $B^2 \le AC$ when $A > 0$ where $B = a_1 b_1 + \cdots + a_n b_n$,

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- Let x be in the field, and consider $Ax^2 + 2Bx + C$

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  as required.
- There are many different proofs of this.

Absolute Values

- Before we can discuss anything connected with convergence we need to know what we mean by “small”, or to be more precise we need to have some measure of the size of a number. The standard way for real numbers is as follows.

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- **Definition 2.5. Absolute Value.** Let x be an element of an ordered field. Then we define the absolute value, or modulus, of x by

$$|x| = \begin{cases} x & \text{when } x \geq 0, \\ -x & \text{when } x < 0. \end{cases}$$

- **Example 2.8**

$$|-\pi| = \pi, \left|\frac{3}{2}\right| = \frac{3}{2}, |0| = 0.$$

Absolute Values

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- 2. For every x we have $|x| \ge 0$.
- 3. For every x we have $|-x| = |x|$. To see this, separate out the three cases $x > 0$, $x = 0$, $x < 0$. When $x = 0$ we have $|-x| = |0| = 0 = |0| = |x|$. When $x > 0$ we have $-x < 0$ and so $|-x| = -(-x) = x = |x|$ and when $x < 0$ we have $-x > 0$ so that $|-x| = -x = |x|$.

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- 4. $a < 0$, $b < 0$. Then $ab > 0$ and

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Suppose that $b \neq 0$. *Then*

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- Since $b \neq 0$ we have $|b| \neq 0$ and so we can divide both sides by $|b|$.

Now we come to something we will use all the time.

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- But by the definition of absolute value we have

$$\begin{aligned} |x+y|^2 = (x+y)^2 &= x^2 + 2xy + y^2 \\ &\le x^2 + |2xy| + y^2 = |x|^2 + 2|x||y| + |y|^2 \\ &= (|x| + |y|)^2. \end{aligned}$$

- **Example 2.9.**

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- Interchanging t and u gives $|u| - |t| \le |u-t| = |t-u|$.
- But one of $|t| - |u|$ and $|u| - |t| = -(|t| - |u|)$ is non-negative, so is

$$= \big||t| - |u|\big|.$$

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$$\mathcal{A} = (-5, 2).$$

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- Hence the complete solution is $x=-4$ or 2.

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- In place of ordered pairs we should, at least initially think of real numbers as being infinite sets of rational numbers.

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- The set $\mathcal{S}$ of Example 2.12 is bounded below and bounded. The set $\mathbb{N}$ is unbounded (presumably - later we will prove this).

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- Hence the set $\mathcal{A}$ is bounded with 1 as a lower bound and 2 as an upper bound..

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- The job of the axiom we are missing is to ensure that there is always a smallest such number.

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- *Proof.* Suppose that $s_1 < s_2$ are two different suprema of $\mathcal{A}$. By the definition of supremum we have $a \leq s_1$ for every $a \in \mathcal{A}$ and so s_2 could not be a least upper bound.

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- Suppose on the contrary that there is a $t > -s$ such that t is a lower bound for $\mathcal{B}$. i.e. for every $b \in \mathcal{B}$. Then $-b \le -t < s$. Thus $-t$ would be a lower upper bound for $\mathcal{A}$ than its supremum s which is absurd.

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Suppose that $\mathcal{A}$ is a non-empty set of real numbers which is bounded above, $y > 0$ and $\mathcal{B} = \{ya : a \in \mathcal{A}\}$. Then $\sup \mathcal{B}$ exists and $\sup \mathcal{B} = y \sup \mathcal{A}$.

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- *Proof.* Since $\mathcal{A}$ is non-empty, so is $\mathcal{B}$.
- Moreover if H is an upper bound for $\mathcal{A}$, then yH is an upper bound for $\mathcal{B}$.
- Hence $s = \sup \mathcal{A}$ and $t = \sup \mathcal{B}$ both exist.
- Moreover sy will be an upper bound for $\mathcal{B}$ and t/y will be an upper bound for $\mathcal{A}$.

- Before moving on to study the properties of the real numbers we just give an inkling of how it is possible to pull over to $\mathbb{R}$ the various axioms which are satisfied by $\mathbb{Q}$

Theorem 11

Suppose that $\mathcal{A}$ is a non-empty set of real numbers which is bounded above, $y > 0$ and $\mathcal{B} = \{ya : a \in \mathcal{A}\}$. Then $\sup \mathcal{B}$ exists and $\sup \mathcal{B} = y \sup \mathcal{A}$.

- *Proof.* Since $\mathcal{A}$ is non-empty, so is $\mathcal{B}$.
- Moreover if H is an upper bound for $\mathcal{A}$, then yH is an upper bound for $\mathcal{B}$.
- Hence $s = \sup \mathcal{A}$ and $t = \sup \mathcal{B}$ both exist.
- Moreover sy will be an upper bound for $\mathcal{B}$ and t/y will be an upper bound for $\mathcal{A}$.
- Hence $t \le sy$ and $s \le t/y \le s$, whence $t = ys$.