Introduction to Analysis

Introduction

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- The original model for this is Euclid's axiomatisation of geometry about 300BC.
- That is the establishment of a few simple basic statements (axioms) from which all propositions are deduced by basic rules of logical deduction.
- The wisdom of Euclids original choice is demonstrated by the observation that in the intervening 2300 years nobody has found anything self contradictory in the vast panoply of geometric theorems which have been established in Euclidean geometry.

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- Nevertheless they can be described by adjusting the axiom which deals with the concept of parallel lines.

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- They had some understanding of positive whole numbers, but, at least in Europe, the systems for describing them derived from the Etruscans was similar to, and eventually evolved into, the Roman numeral system.
- We know how clumsy that is for doing arithemtic, and it is no surprise to learn that there was no simple way of deal even with quite simple fractions.

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- In other words given a particular unit length he understands how to produce line segments whose length is twice, thrice, and so on, the unit length.
- He also understands how to product a line segment whose length ℓ satisfies

  $$\ell : 1 :: m : n$$

  where m and n are positive whole numbers, and which in modern notation is simply

  $$\ell = \frac{\ell}{1} = \frac{m}{n}.$$

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- Yet they could construct such lengths from right angled triangles.

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- That is we just showed that m and n do have a common prime factor contradicting our basic assumption.

- The problem for the Pythagoreans was that this seemed to imply that $\sqrt{2}$ does not exist, and gave a paradox against Pythagoras' theorem. Our problem is to resolve this.

- Of course we are all familiar with the fact that we can get good approximations to $\sqrt{2}$

$$
\begin{aligned}
(1.4)^2 &= 1.96 & (1.5)^2 &= 2.25 \\
(1.41)^2 &= 1.9881 & (1.42)^2 &= 2.0164 \\
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- What this suggests, at least philosophically, is that perhaps we should think of $\sqrt{2}$ as being an infinite collection of rational numbers.

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- The symbol $\in$ is a variant of the Greek epsilon, ϵ or ε but should not be confused with them and one should try to distinguish them when writing them.

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$$\mathcal{S} = \{1, 3, \pi, 7/2, \sqrt{17}\},$$

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- **2.** By some kind of defining formula.

$$\mathcal{T} = \{x : 1 < x < 2\}, \tag{2.1}$$

$$\mathcal{U} = \{(x, y) : x^2 + y^2 = 1\}. \tag{2.2}$$

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- **Example 1.1.**
$$\{x : x^2 < 0\} = \emptyset.$$
- There is an important logical observation. Since the set has no elements its elements can have any property. For example they can be both positive and negative!

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- **Example 1.2.** The set $\mathcal{T} = \{1, 3, \pi\}$ has subsets
$$\{1,3,\pi\},$$
$$\{1,3\},\ \{1,\pi\},\ \{3,\pi\},$$
$$\{1\},\ \{3\},\ \{\pi\},$$
$$\emptyset$$

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- Generally a finite set with k elements has 2^k subsets and
$$\binom{k}{j}$$
subsets with exactly j elements.

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- **Definition 1.4.** The *intersection* of two sets $\mathcal{A}$ and $\mathcal{B}$ is the set which contains the elements which are in both $\mathcal{A}$ and $\mathcal{B}$.
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- **Example 1.4.** In the above example $\mathcal{A} = \{1, 2, 3\}$, $\mathcal{B} = \{2, 3, 4\}$,
$$\mathcal{A} \cap \mathcal{B} = \{2, 3\}.$$
- **Another Example 1.5.**
$$\mathcal{U} = \{x : 0 < x < 1\}, \mathcal{V} = \{1 \le x \le 2\}, \mathcal{U} \cap \mathcal{V} = \emptyset.$$

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- **Definition 1.5.** The *complement* of $\mathcal{B}$ with respect to $\mathcal{A}$ is the set of x in $\mathcal{A}$ which are not in $\mathcal{B}$,

$$\mathcal{A} \setminus \mathcal{B} = \{x : x \in \mathcal{A} \text{ and } x \notin \mathcal{B}\}.$$

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- **Example 1.6.** Again in the example $\mathcal{A} = \{1, 2, 3\}$, $\mathcal{B} = \{2, 3, 4\}$,

$$\mathcal{A} \setminus \mathcal{B} = \{1\}, \quad \mathcal{B} \setminus \mathcal{A} = \{4\}.$$

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- **Example 1.7.** In general

$$(\mathcal{C} \setminus \mathcal{D}) \cap (\mathcal{D} \setminus \mathcal{C}) = \emptyset.$$

and

$$\mathcal{C} \cap (\mathcal{D} \cup \mathcal{E}) = (\mathcal{C} \cap \mathcal{D}) \cup (\mathcal{C} \cap \mathcal{E}).$$

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- Returning to the penultimate example.

$\mathcal{C}$	$\mathcal{D}$	$\mathcal{C}\setminus\mathcal{D}$	$\mathcal{D}\setminus\mathcal{C}$	$(\mathcal{C}\setminus\mathcal{D})\cap(\mathcal{D}\setminus\mathcal{C})$	$\emptyset$
1	1	0	0	0	0
1	0	1	0	0	0
0	1	0	1	0	0
0	0	0	0	0	0

Sets

Here is the other Example.

$$\mathcal{C} \cap (\mathcal{D} \cup \mathcal{E}) = (\mathcal{C} \cap \mathcal{D}) \cup (\mathcal{C} \cap \mathcal{E}).$$

$\mathcal{C}$	$\mathcal{D}$	$\mathcal{E}$	$\mathcal{D} \cup \mathcal{E}$	$\mathcal{C} \cap \mathcal{D}$	$\mathcal{C} \cap \mathcal{E}$	*LHS*	*RHS*
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
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Integers and Rationals

- We have already introduced the standard notation

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\} \quad \text{The natural numbers,}$$

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- We start with the simplest of these sets, $\mathbb{N}$.

Definition 1.6. The Peano axioms for $\mathbb{N}$

The Natural Numbers

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- **4.** If $\mathcal{S}$ is a set with the properties that (a) $1 \in \mathcal{S}$ and (b) whenever $n \in \mathcal{S}$ we have $n+1 \in \mathcal{S}$, then $\mathcal{S} = \mathbb{N}$.

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- Given m let $\mathcal{S}$ denote the set of n for which $m+n \in \mathbb{N}$. Then by Axiom 1 $m+1 \in \mathbb{N}$, so $1 \in \mathcal{S}$. Suppose $n \in \mathcal{S}$. Then $m+n \in \mathbb{N}$ and so by Axiom 1 once more we have $m+n+1 \in \mathbb{N}$ and so $n+1 \in \mathcal{S}$. Hence, by Axiom 4 we have $\mathcal{S} = \mathbb{N}$.

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- In this kind of way various other properties of $\mathbb{N}$ can be established. For example if $m, n \in \mathbb{N}$, then $m+n = n+m$.
- We can also define multiplication by taking $n \times 1 = 1 \times n = n$ and $n \times (m+1) = (n \times m) + n$ and using induction. We can then combine addition and multiplication more generally to show that

$$\ell \times (m+n) = (\ell \times m) + (\ell \times n).$$

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- Later, when developing the ideas of limits we will need to look at the elements of $\mathbb{N}$ in more detail.

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- To avoid this we follow a different route.
- One of the more powerful techniques we have is the ability to create more complex and richer systems out of simpler ones.
- Thus we can think about “extending $\mathbb{N}$ to give $\mathbb{Z}$, and there is a very nice way of doing this by the use of ordered pairs of natural numbers (m, n) and something called equivalence classes.

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- If we use $\mathcal{A}(m, n)$ to denote the set of ordered pairs equivalent to (m, n), then we can define addition and multiplication by

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- In other words we found a way of *constructing* the integers from the natural numbers.

The Rational Numbers

- We can then use a similar procedure to construct the rational numbers by now looking at equivalence classes of ordered pairs (p, q) of integers p, q with $q \neq 0$. For example, let $\mathcal{B}(r, s)$ be the set of such ordered pairs (p, q) with $ps = rq$.

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  and again check that this results in the properties we expect of elements of $\mathbb{Q}$.
- Again I do not want to spend time checking this. The main problem at hand at this stage is dealing with the question of numbers such as $\sqrt{2}$ where something more profound is needed.