Introduction to Analysis: Review of Calculus

Robert C. Vaughan

November 28, 2023

- This course has relatively little to do directly with calculus.

- This course has relatively little to do directly with calculus.
- Analysis was largely developed in the nineteenth century by the need to understand what was meant by a limit and place the fundamental theorems on limits on a sound axiomatic basic, as had been done by Euclid and the Pythagorean school about 2000 years earlier for geometry.

- This course has relatively little to do directly with calculus.
- Analysis was largely developed in the nineteenth century by the need to understand what was meant by a limit and place the fundamental theorems on limits on a sound axiomatic basic, as had been done by Euclid and the Pythagorean school about 2000 years earlier for geometry.
- We shall see that the idea of a limit is intimately connected with what we mean by a number.

- The immediate connection with calculus is that the derivative and integral are usually both defined as a result of a limiting process.

- The immediate connection with calculus is that the derivative and integral are usually both defined as a result of a limiting process.
- Thus given a real valued function f it is usual to define the derivative f' by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

or if we think in terms of a curve $y = f(x)$ in the x, y plane, then

$$\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x}$$

where $\delta y = y(x + \delta x) - y(x)$.

- The immediate connection with calculus is that the derivative and integral are usually both defined as a result of a limiting process.
- Thus given a real valued function f it is usual to define the derivative f' by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

  or if we think in terms of a curve $y = f(x)$ in the x, y plane, then

$$\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x}$$

  where $\delta y = y(x + \delta x) - y(x)$.
- These are, of course, just symbolic representations of the same thing.

- Example.
  $f(x) = x^4$.

- Example.
  $f(x) = x^4$.
- $$f(x+h)-f(x) = (x+h)^4 - x^4 = 4x^3h + 6x^2h^2 + 4xh^3 + h^4,$$

- Example.
  $f(x) = x^4$.
- $$f(x+h)-f(x) = (x+h)^4 - x^4 = 4x^3h + 6x^2h^2 + 4xh^3 + h^4,$$
- $$\frac{f(x+h)-f(x)}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3$$

- Example.
  $f(x) = x^4$.
- $$f(x+h)-f(x) = (x+h)^4 - x^4 = 4x^3h + 6x^2h^2 + 4xh^3 + h^4,$$
- $$\frac{f(x+h)-f(x)}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3$$
- Then each of the terms $6x^2h$, $4xh^2$, h^3 will tend to 0 as h tends to 0 and we will conclude that in this case
  $$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 4x^3.$$

- In calculus we learn, or at least see, many formulæ for derivates. Maybe you can help me out by recalling some of them.

- In calculus we learn, or at least see, many formulæ for derivates. Maybe you can help me out by recalling some of them.
- Let $f(x) = x^n$ (n a non-negative whole number). What is $f'(x)$?

- In calculus we learn, or at least see, many formulæ for derivates. Maybe you can help me out by recalling some of them.
- Let $f(x) = x^n$ (n a non-negative whole number). What is $f'(x)$?
- What about $f(x) = x^{-1}$?

- In calculus we learn, or at least see, many formulæ for derivates. Maybe you can help me out by recalling some of them.
- Let $f(x) = x^n$ (n a non-negative whole number). What is $f'(x)$?
- What about $f(x) = x^{-1}$?
- Or $f(x) = x^{-n}$?

- In calculus we learn, or at least see, many formulæ for derivates. Maybe you can help me out by recalling some of them.
- Let $f(x) = x^n$ (n a non-negative whole number). What is $f'(x)$?
- What about $f(x) = x^{-1}$?
- Or $f(x) = x^{-n}$?
- Or $f(x) = x^{1/2}$?

- In calculus we learn, or at least see, many formulæ for derivates. Maybe you can help me out by recalling some of them.
- Let $f(x) = x^n$ (n a non-negative whole number). What is $f'(x)$?
- What about $f(x) = x^{-1}$?
- Or $f(x) = x^{-n}$?
- Or $f(x) = x^{1/2}$?
- Let $f(x) = \sin x$. What is $f'(x)$?

- In calculus we learn, or at least see, many formulæ for derivates. Maybe you can help me out by recalling some of them.
- Let $f(x) = x^n$ (n a non-negative whole number). What is $f'(x)$?
- What about $f(x) = x^{-1}$?
- Or $f(x) = x^{-n}$?
- Or $f(x) = x^{1/2}$?
- Let $f(x) = \sin x$. What is $f'(x)$?
- Let $f(x) = \arctan x$. What is $f'(x)$?

- In practice the functions we come across are much more complicated than this, so it is normal to develop some general formulæ.

- In practice the functions we come across are much more complicated than this, so it is normal to develop some general formulæ.
- The simplest is the sum of two general functions, suppose f, g h are connected by $f(x) = g(x) + h(x)$. Then

$$f'(x) = g'(x) + h'(x).$$

One way of thinking of this is that differentiation is a "linear operator".

- In practice the functions we come across are much more complicated than this, so it is normal to develop some general formulæ.
- The simplest is the sum of two general functions, suppose f, g h are connected by $f(x) = g(x) + h(x)$. Then

$$f'(x) = g'(x) + h'(x).$$

  One way of thinking of this is that differentiation is a "linear operator".
- How about a product, $f(x) = g(x)h(x)$? Is there a formula for

- In practice the functions we come across are much more complicated than this, so it is normal to develop some general formulæ.
- The simplest is the sum of two general functions, suppose f, g h are connected by $f(x) = g(x) + h(x)$. Then

$$f'(x) = g'(x) + h'(x).$$

  One way of thinking of this is that differentiation is a "linear operator".
- How about a product, $f(x) = g(x)h(x)$? Is there a formula for
- How about $f(x) = g(h(x))$?

- In practice the functions we come across are much more complicated than this, so it is normal to develop some general formulæ.
- The simplest is the sum of two general functions, suppose f, g h are connected by $f(x) = g(x) + h(x)$. Then

$$f'(x) = g'(x) + h'(x).$$

  One way of thinking of this is that differentiation is a "linear operator".
- How about a product, $f(x) = g(x)h(x)$? Is there a formula for
- How about $f(x) = g(h(x))$?
- $f(x) = 1/h(x)$?

- Example

$$y = (x^2 + 2)^{10}.$$

- Now we come to tricky one. Let

$$f(x) = \frac{g(x)}{h(x)}.$$

what is $f'(x)$?

A tricky example

- Now define

$$f(x) = \begin{cases} \sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$$g(x) = \begin{cases} x\sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$$h(x) = \begin{cases} x^2\sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

A tricky example

- Now define

$$f(x) = \begin{cases} \sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$$g(x) = \begin{cases} x\sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$$h(x) = \begin{cases} x^2\sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

- If $x \neq 0$, then each of these is differentiable, by use of the chain product rules. For example

$$h'(x) = 2x\sin(1/x) - cos(1/x) \quad (x \neq 0).$$

A tricky example

- Now define

$$f(x) = \begin{cases} \sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$$g(x) = \begin{cases} x\sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$$h(x) = \begin{cases} x^2\sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

- If $x \neq 0$, then each of these is differentiable, by use of the chain product rules. For example

$$h'(x) = 2x\sin(1/x) - cos(1/x) \quad (x \neq 0).$$

- What happens in each case when $x = 0$?

A tricky example

- Now define

$$f(x) = \begin{cases} \sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$$g(x) = \begin{cases} x\sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$$h(x) = \begin{cases} x^2\sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

- If $x \neq 0$, then each of these is differentiable, by use of the chain product rules. For example

$$h'(x) = 2x\sin(1/x) - cos(1/x) \quad (x \neq 0).$$

- What happens in each case when $x = 0$?
- How could we be sure that our guesses are correct?

$$
\begin{aligned}
y &= x^n, & y' &= nx^{n-1},\\
y &= \sin(x), & y' &= \cos(x),\\
y &= \cos(x), & y' &= -\sin(x),\\
y &= \tan(x), & y' &= \sec^2(x),\\
y &= e^x, & y' &= e^x,\\
y &= \ln(x), & y' &= 1/x,\\
y &= \arctan(x), & y' &= \frac{1}{1+x^2},\\
y &= \arcsin(x), & y' &= (1-x^2)^{-1/2},\\
y &= u+v, & y' &= u'+v',\\
y &= uv, & y' &= u'v+uv',\\
y &= \frac{u}{v}, & y' &= \frac{u'v-uv'}{v^2}.
\end{aligned}
$$

Inverse Functions

- Suppose that the function

$$y = f(x)$$

is such that each y corresponds to a unique x.

Inverse Functions

- Suppose that the function

$$y = f(x)$$

is such that each y corresponds to a unique x.

- Then for each y we can define a function

$$g(y) = x$$

which has the property that $g(f(x)) = x$. Such a function is called an inverse function.

Inverse Functions

- Suppose that the function
$$y = f(x)$$
is such that each y corresponds to a unique x.
- Then for each y we can define a function
$$g(y) = x$$
which has the property that $g(f(x)) = x$. Such a function is called an inverse function.
- Now we can apply the chain rule and obtain
$$g'(f(x))f'(x) = 1.$$

Inverse Functions

- Suppose that the function

$$y = f(x)$$

is such that each y corresponds to a unique x.

- Then for each y we can define a function

$$g(y) = x$$

which has the property that $g(f(x)) = x$. Such a function is called an inverse function.

- Now we can apply the chain rule and obtain

$$g'(f(x))f'(x) = 1.$$

- In other words $g'(y) = \dfrac{1}{f'(x)}$.

Inverse Functions

- Suppose that the function
$$y = f(x)$$
is such that each y corresponds to a unique x.
- Then for each y we can define a function
$$g(y) = x$$
which has the property that $g(f(x)) = x$. Such a function is called an inverse function.
- Now we can apply the chain rule and obtain
$$g'(f(x))f'(x) = 1.$$
- In other words $g'(y) = \dfrac{1}{f'(x)}$.
- An important example is $y = f(x) = e^x$ and $g(y) = \ln y$. Then $f'(x) = e^x = y$, so that $g(y) = 1/f'(x) = 1/y$.

Some examples

- 1.

$$y = x^5 - x^4 + x^2$$
$$y' = 5x^4 - 4x^3 + 2x.$$

Some examples

- 1.

$$y = x^5 - x^4 + x^2$$
$$y' = 5x^4 - 4x^3 + 2x.$$

- 2.

$$y = x^2 \sin x$$
$$y' = 2x \sin x + x^2 \cos x.$$

Some examples

- 1.

$$y = x^5 - x^4 + x^2$$
$$y' = 5x^4 - 4x^3 + 2x.$$

- 2.

$$y = x^2 \sin x$$
$$y' = 2x \sin x + x^2 \cos x.$$

- 2.

$$y = \frac{1-x}{1+x}$$
$$y' = \frac{(1+x)(-1) - (1-x).1}{(1+x)^2} = \frac{-2}{(1+x)^2}.$$

Equation of the tangent

- It is sometimes useful in applications to know the tangent to a point $(x_0, f(x_0))$ on a curve $y = f(x)$.

Equation of the tangent

- It is sometimes useful in applications to know the tangent to a point $(x_0, f(x_0))$ on a curve $y = f(x)$.
- The derivative at the point is $f'(x_0)$ and is the slope of the curve at that point, so will also be the slope of the tangent.

Equation of the tangent

- It is sometimes useful in applications to know the tangent to a point $(x_0, f(x_0))$ on a curve $y = f(x)$.
- The derivative at the point is $f'(x_0)$ and is the slope of the curve at that point, so will also be the slope of the tangent.
- The equation of a line through the point (x_0, y_0) with slope m is given by

$$y - y_0 = m(x - x_0).$$

substituting from above gives

$$y - f(x_0) = f'(x_0)(x - x_0)$$

which we can rearrange to give

$$y = f'(x_0)x + f(x_0) - x_0 f'(x_0).$$

Implicit differentiation

- Something which is sometimes useful but often isn't covered in calculus courses is implicit differentiation.

Implicit differentiation

- Something which is sometimes useful but often isn't covered in calculus courses is implicit differentiation.
- Suppose that there is no simple formula connection x and $y = f(x)$. For example $x^3 + y^3 - 2x + 3y = 0$.

Implicit differentiation

- Something which is sometimes useful but often isn't covered in calculus courses is implicit differentiation.
- Suppose that there is no simple formula connection x and $y = f(x)$. For example $x^3 + y^3 - 2x + 3y = 0$.
- Obviously we can differentiate both sides and obtain

$$3x^2 + 3y^2\frac{dy}{dx} - 2 + 3\frac{dy}{dx} = 0.$$

Implicit differentiation

- Something which is sometimes useful but often isn't covered in calculus courses is implicit differentiation.
- Suppose that there is no simple formula connection x and $y = f(x)$. For example $x^3 + y^3 - 2x + 3y = 0$.
- Obviously we can differentiate both sides and obtain

$$3x^2 + 3y^2 \frac{dy}{dx} - 2 + 3\frac{dy}{dx} = 0.$$

- Then we can solve for the derivative to obtain

$$\frac{dy}{dx} = \frac{2 - 3x^2}{3y^2 + 3}.$$

Implicit differentiation

- Something which is sometimes useful but often isn't covered in calculus courses is implicit differentiation.
- Suppose that there is no simple formula connection x and $y = f(x)$. For example $x^3 + y^3 - 2x + 3y = 0$.
- Obviously we can differentiate both sides and obtain

$$3x^2 + 3y^2\frac{dy}{dx} - 2 + 3\frac{dy}{dx} = 0.$$

- Then we can solve for the derivative to obtain

$$\frac{dy}{dx} = \frac{2 - 3x^2}{3y^2 + 3}.$$

- The point $(2, -1)$ lies on the curve. Thus

$$\frac{dy}{dx}_{(x,y)=(2,-1)} = -\frac{5}{3}$$

and the tangent at $(2, -1)$ is given by $y = -\frac{5}{3}x + \frac{7}{3}$.

- If a curve does not have slope 0 at a particular point, then it cannot be a local maximum or minimum.

- If a curve does not have slope 0 at a particular point, then it cannot be a local maximum or minimum.
- Thus to find candidate extremal points we can first check for points for which the derivative is 0.

- If a curve does not have slope 0 at a particular point, then it cannot be a local maximum or minimum.
- Thus to find candidate extremal points we can first check for points for which the derivative is 0.
- That is not a guarantee. The point might be a point of inflexion.

- If a curve does not have slope 0 at a particular point, then it cannot be a local maximum or minimum.
- Thus to find candidate extremal points we can first check for points for which the derivative is 0.
- That is not a guarantee. The point might be a point of inflexion.
- However if $f'(x) = 0$ when $x = x_0$ **AND** $f'(x) > 0$ **just to the left of** x_0 **and** $f'(x) < 0$ **just to the right of** x_0**, then** f **must have a local maximum at** x_0**.**

- If a curve does not have slope 0 at a particular point, then it cannot be a local maximum or minimum.
- Thus to find candidate extremal points we can first check for points for which the derivative is 0.
- That is not a guarantee. The point might be a point of inflexion.
- However if $f'(x) = 0$ when $x = x_0$ **AND** $f'(x) > 0$ **just to the left of** x_0 **and** $f'(x) < 0$ **just to the right of** x_0**, then** f **must have a local maximum at** x_0**.**
- If these inequalities are reversed, then we have a local minimum.

- Example. $y = x^3 - x$. Then $\frac{dy}{dx} = 3x^2 - 1$ and this is 0 when $x = \frac{\pm 1}{\sqrt{3}}$.

- Example. $y = x^3 - x$. Then $\frac{dy}{dx} = 3x^2 - 1$ and this is 0 when $x = \frac{\pm 1}{\sqrt{3}}$.
- If $x < -1/\sqrt{3}$, then $x^2 > 1/3$, $\frac{dy}{dx} > 0$, and if $-1/\sqrt{3} < x < 0$, then $x^2 < 1/3$ and so $\frac{dy}{dx} < 0$.

- Example. $y = x^3 - x$. Then $\frac{dy}{dx} = 3x^2 - 1$ and this is 0 when $x = \frac{\pm 1}{\sqrt{3}}$.
- If $x < -1/\sqrt{3}$, then $x^2 > 1/3$, $\frac{dy}{dx} > 0$, and if $-1/\sqrt{3} < x < 0$, then $x^2 < 1/3$ and so $\frac{dy}{dx} < 0$.
- Thus $x^3 - x$ has a local maximum at $-1/\sqrt{3}$.

- Example. $y = x^3 - x$. Then $\frac{dy}{dx} = 3x^2 - 1$ and this is 0 when $x = \frac{\pm 1}{\sqrt{3}}$.
- If $x < -1/\sqrt{3}$, then $x^2 > 1/3$, $\frac{dy}{dx} > 0$, and if $-1/\sqrt{3} < x < 0$, then $x^2 < 1/3$ and so $\frac{dy}{dx} < 0$.
- Thus $x^3 - x$ has a local maximum at $-1/\sqrt{3}$.
- These inequalities are reversed in the neighbourhood of $1/\sqrt{3}$, so $x^3 - x$ has a local minimum at $1/\sqrt{3}$.

- Example. $y = x^3 - x$. Then $\frac{dy}{dx} = 3x^2 - 1$ and this is 0 when $x = \frac{\pm 1}{\sqrt{3}}$.
- If $x < -1/\sqrt{3}$, then $x^2 > 1/3$, $\frac{dy}{dx} > 0$, and if $-1/\sqrt{3} < x < 0$, then $x^2 < 1/3$ and so $\frac{dy}{dx} < 0$.
- Thus $x^3 - x$ has a local maximum at $-1/\sqrt{3}$.
- These inequalities are reversed in the neighbourhood of $1/\sqrt{3}$, so $x^3 - x$ has a local minimum at $1/\sqrt{3}$.
- Alternatively one can check $\frac{d^2y}{dx^2} = 6x$. At $x = \frac{\pm 1}{\sqrt{3}}$ this is $\pm 2\sqrt{3}$.

- Example. $y = x^3 - x$. Then $\frac{dy}{dx} = 3x^2 - 1$ and this is 0 when $x = \frac{\pm 1}{\sqrt{3}}$.
- If $x < -1/\sqrt{3}$, then $x^2 > 1/3$, $\frac{dy}{dx} > 0$, and if $-1/\sqrt{3} < x < 0$, then $x^2 < 1/3$ and so $\frac{dy}{dx} < 0$.
- Thus $x^3 - x$ has a local maximum at $-1/\sqrt{3}$.
- These inequalities are reversed in the neighbourhood of $1/\sqrt{3}$, so $x^3 - x$ has a local minimum at $1/\sqrt{3}$.
- Alternatively one can check $\frac{d^2y}{dx^2} = 6x$. At $x = \frac{\pm 1}{\sqrt{3}}$ this is $\pm 2\sqrt{3}$.
- Note that sometimes computing the second derivative can be a right pain and the above method is usually easier.

- Armed with our understanding of derivatives it is often useful in applications to sketch a relevant curve.

- Armed with our understanding of derivatives it is often useful in applications to sketch a relevant curve.
- Example. $y = x^3(x-1)^2$.

- Armed with our understanding of derivatives it is often useful in applications to sketch a relevant curve.
- Example. $y = x^3(x-1)^2$.
- Second Example.

$$y = \frac{1}{x^2 + x + 1}.$$

- Armed with our understanding of derivatives it is often useful in applications to sketch a relevant curve.
- Example. $y = x^3(x-1)^2$.
- Second Example.

$$y = \frac{1}{x^2 + x + 1}.$$

- Third example.

$$y = \frac{x^2 - 1}{x(x+2)}.$$