MATH 401 INTRODUCTION TO ANALYSIS, SPRING TERM 2024, SOLUTIONS 11

1. Decide the convergence of the each of the following series, in each case proving your assertion (i) $\sum_{n=1}^{\infty} \frac{2}{n^3+1}$, (ii) $\sum_{n=1}^{\infty} \frac{3}{2n+1}$, (iii) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n-1)!} 3^n$, (iv) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n-1)!} 5^n$, (v) $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/3}$, (vi) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)(-1)^n$.

(i) converges by the comparison test since $3/(n^3+2) \le 3n^{-2}$ and $\sum_{n=1}^{\infty} n^{-2}$ converges. (ii) diverges by the comparison test since $3/(2n+1) \ge n^{-1}$ and $\sum_{n=1}^{\infty} n^{-1}$ diverges. (iii) converges by the ratio test as $\lim_{n\to\infty} |a_{n+1}/a_n| = \lim_{n\to\infty} (n+1)^2(2n+1)^{-1}(2n)^{-1} = 3/4 < 1$. (iv) diverges by the ratio test as $\lim_{n\to\infty} |a_{n+1}/a_n| = 5/4 > 1$. (v) $n^{-1/3}$ is a decreasing sequence tending to 0. Hence the series converges by the Liebnitz test. (vi) diverges since the general term does not tend to 0.

2. Prove that $\sum_{n=1}^{\infty} x^n \frac{(n!)^3}{(3n)!}$ converges when $|x| < 27$ and diverges when $|x| > 27$.

The ratio $|a_{n+1}/a_n| = |x|(n+1)^3(3n+1)^{-1}(3n+2)^{-1}(3n+3)^{-1} \to |x|/27$ as $n \to \infty$. Hence the series converges when $|x| < 27$ and diverges when $|x| > 27$.

3. Prove that $\sum_{n=1}^{\infty} x^n \frac{(-1)^{(n-1)}}{(2n-1)!}$ converges for all real x.

The ratio test gives $|x|(2n)^{-1}(2n+1)^{-1} \to 0$ as $n \to \infty$. Hence the series converges for all real x.

4. This question uses the notation and results of homework 10. (i) Prove that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges, and let e be its value. (ii) Prove that for $n \in \mathbb{N}$, $a_n = 1 + \sum_{m=1}^{n} \frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{m-1}{n}\right)$ and deduce that $a_n \le e$ and hence that $\lim_{n\to\infty} a_n \le e$. (iii) Suppose $1 \le r \le n$. Prove that $1 + \sum_{m=1}^{r} \frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{m-1}{n}\right) \le a_n$ and deduce that $1 + \sum_{m=1}^{r} \frac{1}{m!} \le \lim_{n\to\infty} a_n$ and hence that $\lim_{n\to\infty} a_n = \lim_{n\to} b_n = e$.

(i) The ratio test has $l = 0$. (ii) By the binomial theorem,

$$a_n = 1 + \sum_{m=1}^{n} \frac{1}{m!} \cdot \frac{n!}{(n-m)!} \cdot \frac{1}{n^m} = 1 + \sum_{m=1}^{n} \frac{1}{m!} \cdot \frac{n(n-1)\ldots(n-m+1)}{n^m}.$$

The general term is $\le 1/m!$, so $a_n \le e$. We know from homework 10 that $\lim_{n\to\infty} a_n$ exists. (iii) Each term in the series in (ii) is non–negative. Hence

$$1 + \sum_{m=1}^{r} \frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{m-1}{n}\right) \le a_n.$$

Let $n \to \infty$ on both sides of this. Thus $1 + \sum_{m=1}^{r} \frac{1}{m!} \le \lim_{n\to\infty} a_n$. Now let $r \to \infty$. Thus $e \le \lim_{n\to\infty} a_n$ and then the final conclusion follows from (ii) and homework 10.