MATH 401 INTRODUCTION TO ANALYSIS-I, SPRING TERM 2024, SOLUTIONS 10

Throughout we define $a_n = (1+1/n)^n$ ($n \in \mathbb{N}$), $b_n = (1-1/n)^{-n}$ ($n = 2, 3, 4, \ldots$).

1. (i) Prove that for $n \in \mathbb{N}$ we have $\left(\frac{n^2+2n}{n^2+2n+1}\right)^{n+1} \geq \frac{n}{n+1}$. (ii) Prove that for $n \in \mathbb{N}$, $\left(\frac{n+2}{n+1}\right)^{n+1} \left(\frac{n}{n+1}\right)^n \geq 1$. (iii) Prove that $\langle a_n \rangle$ is increasing, and $a_n \geq 2$ for every $n \in \mathbb{N}$.

(i) By the binomial inequality $LHS \geq 1 - \frac{n+1}{(n+1)^2} = \frac{n}{n+1}$. (ii) $LHS = \frac{LHS(i)}{RHS(i)} \geq 1$. (iii) $a_{n+1}/a_n = LHS(ii)$.

2. (i) Suppose that $n \geq 2$. Prove that $\left(\frac{n^2}{n^2-1}\right)^{n+1} \geq \frac{n}{n-1}$. (ii) Suppose that $n \geq 2$. Prove that $\left(\frac{n}{n-1}\right)^n \left(\frac{n}{n+1}\right)^{n+1} \geq 1$. (iii) Prove that $\langle b_n \rangle$ is decreasing and $b_n \leq 4$.

(i) By the binomial inequality $LHS \geq 1 + \frac{n+1}{n^2-1} = \frac{n}{n-1}$. (ii) $LHS = \frac{LHS(i)}{RHS(i)} \geq 1$. (iii) $b_n/b_{n+1} = LHS(ii)$.

3. (i) Suppose that $n \geq 2$. Prove that $1 - 1/n \leq a_n/b_n \leq 1$. (ii) Suppose that $n \geq 2$. Deduce that $a_n \leq 4$, $b_n \geq 2$, $\langle a_n \rangle$ converges, $\langle b_n \rangle$ converges, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

(i) We have $\frac{a_n}{b_n} = \left(1 - \frac{1}{n^2}\right)^n$. Thus $a_n/b_n < 1$, and by the binomial inequality $a_n/b_n \geq 1 - 1/n$. (ii) By 3(i) and 2(iii), $a_n \leq b_n \leq 4$ (*). By 3(i) and 1(iii), $b_n \geq a_n \geq 2$ (**). By 1(iii) $\langle a_n \rangle$ is increasing and by (*) is bounded above. By 2(iii) $\langle b_n \rangle$ is decreasing and by (**) is bounded below. Thus both sequences converge and in each case the limit is at least 2. By the combination theorem $\langle a_n/b_n \rangle$ converges to the ratio of the limits. Moreover by 3(i) and the sandwich theorem $\lim_{n\to\infty} a_n/b_n = 1$.