MATH 401 INTRODUCTION TO ANALYSIS, SPRING TERM 2024, SOLUTIONS 7

1. Let $\mathcal{U} = \left\{ \frac{2n+1}{n+1} : n \in \mathbb{N} \right\}$.

(i) Prove that $\mathcal{U}$ is non-empty and bounded above by 2.

(ii) Prove that if a is a real number with $a < 2$, then there is an $n \in \mathbb{N}$ such that $a < \frac{2n+1}{n+1}$.

(iii) Prove that $\sup \mathcal{U} = 2$.

(i) We have $\frac{3}{2} = \frac{2+1}{1+1} \in \mathcal{U}$. Moreover, for every $n \in \mathbb{N}$ we have $\frac{2n+1}{n+1} = 2 - \frac{1}{n+1} < 2$. Hence $\mathcal{U}$ is non–empty and bounded above by 2. (ii) By the Archimedean property there is an $n \in \mathbb{N}$ such that $n > \frac{1}{2-a} - 1$. Hence $n + 1 > \frac{1}{2-a}$, so that $2 - a > \frac{1}{n+1}$ and $\frac{2n+1}{n+1} > a$. (iii) By (i) $\sup \mathcal{U}$ exists, and $\sup \mathcal{U} \leq 2$. By (ii) if $a < 2$, then a cannot be an upper bound for $\mathcal{U}$. Hence 2 is the least upper bound.

2. Prove that for all $n \geq 7$ we have $3^n \leq n!$.

We use induction on n. Base case $n = 7$. LHS $= 3^7 = 2187 < 5040 = 7! =$ RHS. Inductive step. Suppose $n \geq 7$ and $3^n \leq n!$. Then $3^{n+1} \leq (n!).3 \leq (n!)(n+1) = (n+1)!$.

3. Let $a_n = \frac{n+1}{n}$. Prove that $\lim_{n\to\infty} a_n = 1$.

Let $\varepsilon > 0$ and $N = 1/\varepsilon$. Then whenever $n > N$ we have

$$|a_n - 1| = \left| \frac{n+1}{n} - 1 \right| = \left| \frac{1}{n} \right| = \frac{1}{n} < \frac{1}{N} = \varepsilon.$$