MATH 401 INTRODUCTION TO ANALYSIS, SPRING TERM 2024, PROBLEMS 11

Return by Monday 1st April

1. Decide the convergence of the each of the following series, in each case proving your assertion.

$$\text{(i)} \sum_{n=1}^{\infty} \frac{2}{n^3+1} \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{3}{2n+1} \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n-1)!} 3^n$$

$$\text{(iv)} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n-1)!} 5^n \quad \text{(v)} \sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/3} \quad \text{(vi)} \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)(-1)^n.$$

2. Prove that

$$\sum_{n=1}^{\infty} x^n \frac{(n!)^3}{(3n)!}$$

converges when $|x| < 27$ and diverges when $|x| > 27$.

3. Prove that

$$\sum_{n=1}^{\infty} x^n \frac{(-1)^{(n-1)}}{(2n-1)!}$$

converges for all real x.

4. This question uses the notation and results of homework 10. (i) Prove that

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges, and let e be its value.
(ii) Prove that for $n \in \mathbb{N}$,

$$a_n = 1 + \sum_{m=1}^{n} \frac{1}{m!} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)$$

and deduce that $a_n \le e$ and hence that $\lim_{n\to\infty} a_n \le e$.
(iii) Suppose $1 \le r \le n$. Prove that

$$1 + \sum_{m=1}^{r} \frac{1}{m!} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \le a_n$$

and deduce that

$$1 + \sum_{m=1}^{r} \frac{1}{m!} \le \lim_{n\to\infty} a_n$$

and hence that $\lim_{n\to\infty} a_n = \lim_{n\to} b_n = e$.