MATH 401 INTRODUCTION TO ANALYSIS-I, SPRING TERM 2024, PROBLEMS 10

Return by Monday 25th March

The questions below are interconnected. In at least three places the binomial inequality

$$(1+x)^n \geq 1 + nx$$

valid whenever $n \in \mathbb{N}$ and $x \geq -1$ will be useful. Throughout we define $a_n = (1+1/n)^n$ $(n \in \mathbb{N})$, $b_n = (1-1/n)^{-n}$ $(n = 2, 3, 4, \dots)$.

1. (i) Prove that for $n \in \mathbb{N}$ we have

$$\left(\frac{n^2+2n}{n^2+2n+1}\right)^{n+1} \geq \frac{n}{n+1}.$$

(ii) Prove that for $n \in \mathbb{N}$,

$$\left(\frac{n+2}{n+1}\right)^{n+1} \left(\frac{n}{n+1}\right)^{n} \geq 1.$$

(iii) Prove that $\langle a_n \rangle$ is increasing, and $a_n \geq 2$ for every $n \in \mathbb{N}$.

2. (i) Suppose that $n \geq 2$. Prove that

$$\left(\frac{n^2}{n^2-1}\right)^{n+1} \geq \frac{n}{n-1}.$$

(ii) Suppose that $n \geq 2$. Prove that

$$\left(\frac{n}{n-1}\right)^{n} \left(\frac{n}{n+1}\right)^{n+1} \geq 1.$$

(iii) Prove that $\langle b_n \rangle$ is decreasing and $b_n \leq 4$.

3. (i) Suppose that $n \geq 2$. Prove that $1 - 1/n \leq a_n/b_n \leq 1$.
(ii) Suppose that $n \geq 2$. Deduce that $a_n \leq 4$, $b_n \geq 2$, $\langle a_n \rangle$ converges, $\langle b_n \rangle$ converges, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.