MATH 401 INTRODUCTION TO ANALYSIS-I, SPRING TERM 2024, PROBLEMS 9

Return by Monday 18th March

1. Let $a_n = (-1)^n$. We have seen in class that $\langle a_n \rangle$ diverges. Define

$$b_n = \frac{a_1 + \cdots + a_n}{n},$$

the "average" of a_n. Prove that $\langle b_n \rangle$ converges.

2. Suppose that $0 < k < 1$ and $\langle x_n \rangle$ satisfies $|x_{n+1}| < k|x_n|$ for $n = 1, 2, 3, \ldots$. Prove that

(i) $|x_n| \leq k^{n-1}|x_1|$,
(ii) $\lim_{n\to\infty} x_n = 0$.

3. Suppose that the sequence $\langle a_n \rangle$ satisfies

$$a_1 = \frac{1}{3},\ a_2 = \frac{1}{2},\ a_n = \frac{1}{2}a_{n-1} + \frac{1}{3}a_{n-2} + \frac{1}{6}\ (n \geq 3).$$

(i) Prove that $\langle a_n \rangle$ is increasing and bounded.
(ii) Prove that $\langle a_n \rangle$ converges and find the limit.