MATH 401 INTRODUCTION TO ANALYSIS, SPRING SEMESTER 2024, PRACTICE FINAL EXAM

The final exam for the course will be on Monday 29th April, 4:40pm-6:30pm, 008 Mueller Lab.

The location for the final exam can be checked at

http://www.campusmaps.psu.edu/print/

1. (25 points) Find all real numbers x such that

$$\big||x+1| - |3x-1|\big| < 1.$$

2. (25 points) Let $\mathcal{A} = \left\{\frac{3+n+n^2}{n^2} : n \in \mathbb{N}\right\}$.
(i) Prove that $\inf \mathcal{A}$ and $\sup \mathcal{A}$ exist.
(ii) Prove that $\inf \mathcal{A} = 1$.
(iii) Prove that $\sup \mathcal{A} = 5$.
(iv) Is $1 \in \mathcal{A}$?

3. (25 points) Prove that $5^{1/3}$ is irrational.

4. (25 points) (i) Prove that if $n \geq 6$, then $2^n \leq (n-1)!$.
(ii) Prove that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

5. (25 points) Prove, using the definition of a limit, that

$$\lim_{n\to\infty} \frac{n^2+n}{n^2+2n+1} = 1.$$

6. (25 points) The sequence $\langle x_n \rangle$ is defined inductively by $x_1 = 3$,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right).$$

(i) Prove that $x_n > 0$ for every $n \in \mathbb{N}$.
(ii) Prove that $x_n^2 \geq 5$ for every $n \in \mathbb{N}$.
(iii) Prove that $\langle x_n \rangle$ is decreasing.
(iv) Prove that $\langle x_n \rangle$ converges and find the limit.

7. (25 points) State in each case the values of $x \in \mathbb{R}$ for which the series converges.

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{1+|x|^n}{2+|x|^n}, \quad \text{(ii)} \quad \sum_{n=1}^{\infty} \frac{|x|}{(n+|x|)}, \quad \text{(iii)} \quad \sum_{n=1}^{\infty} \frac{n^2}{2^n} x^n, \quad \text{(iv)} \quad \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

8. (25 points) Prove that the quartic equation $x^4 - 12x^2 + x + 24 = 0$ has four real roots.