Math 401, Spring Term 2024, Model Solutions to Practice Exam 3

Note: Exam 3 is on Wednesday 10th April, 1:25 in Room 011 Huck.

1. Decide the convergence of the each of the following series, in each case proving your assertion.

$$\text{(i)} \sum_{n=1}^{\infty} \frac{3}{n^3+2} \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{4}{3n+2} \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}(26)^n$$

$$\text{(iv)} \sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}(28)^n \quad \text{(v)} \sum_{n=1}^{\infty}(-1)^{n-1}n^{-1/4} \quad \text{(vi)} \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)(-1)^n.$$

(i) $\frac{3}{n^3+2} < \frac{3}{n^2}$ so converges by comparison. (ii) $\frac{4}{3n+2} > \frac{4/5}{n}$ so diverges by comparison. (iii) Converges by ratio test. (iv) Diverges by ratio test. (v) Converges by Leibnitz. (vi) Diverges since general term does not have limit 0.

2. Prove, using only the definition of limit, that $\lim_{x\to 1}(5x-3) = 2$.

Let $\varepsilon > 0$ and choose $\delta - \varepsilon/5$. Then, whenever $0 < |x-1| < \delta$ we have $|(5x-3)-2| = 5|x-1| < 5\delta = \varepsilon$.

3. Evaluate the following limits.

$$\text{(i)} \lim_{x\to 3} \frac{x^3+5x+7}{x^4+6x^2+8} \quad \text{(ii)} \lim_{x\to 3} \frac{x^2-4x+3}{x^2-2x-3}.$$

(i) $\frac{49}{143}$, (ii) When $x \neq 3$,

$$\frac{x^2-4x+3}{x^2-2x-3} = \frac{x-1}{x+1},$$

so limit is $\frac{1}{2}$.

4. Define $f : (-1,1) \mapsto \mathbb{R} : f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$. Prove that if $\xi \in (-1,1)$, then

$$\lim_{x\to\xi} f(x) = f(\xi).$$

Let $\xi \in (-1,1)$ and $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon(1-|\xi|)}{1+\varepsilon}$. Note that

$$|\xi| + \delta = |\xi| + \frac{\varepsilon - \varepsilon|\xi|}{1+\varepsilon} = \frac{|\xi|+\varepsilon}{1+\varepsilon} < \frac{1+\varepsilon}{1+\varepsilon} = 1.$$

Then, whenever $0 < |x-\xi| < \delta$ we have

$$f(x) - f(\xi) = \sum_{n=1}^{\infty} \frac{(x-\xi)(x^{n-1}+x^{n-2}\xi+\cdots+\xi^{n-1})}{n},$$

$$|f(x)-f(\xi)| \le |x-\xi| \sum_{n=1}^{\infty} \frac{n(|\xi|+\delta)^{n-1}}{n} = \frac{|x-\xi|}{1-|\xi|-\delta} < \frac{\delta}{1-|\xi|-\delta} = \varepsilon.$$