Math 401, Spring Term 2023, Model Solutions to Practice Exam 2

Note: Exam 2 is on Wednesday 13th March, 1:25 in Room 011 Huck.

1. Let $\mathcal{A}$, $\mathcal{B}$ be non–empty sets of real numbers which are bounded above, and let $\mathcal{A}+2\mathcal{B}$ denote the set of numbers of the form $a+2b$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$. (i) Prove that $\sup(\mathcal{A}+2\mathcal{B})$ exists. (ii) Prove that $\sup(\mathcal{A}+2\mathcal{B}) \leq \sup\mathcal{A} + 2\sup\mathcal{B}$. (iii) Let $\delta > 0$. Prove that there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a > \sup\mathcal{A} - \delta$ and $b > \sup\mathcal{B} - \delta$. (iv) Deduce that $\sup(\mathcal{A}+2\mathcal{B}) = \sup\mathcal{A} + 2\sup B$.

(i) $\mathcal{A}$ and $\mathcal{B}$ are non–empty, so there exists an $a \in \mathcal{A}$ and a b in $\mathcal{B}$. Hence $a+2b \in \mathcal{A}+2\mathcal{B}$ so $\mathcal{A}+2\mathcal{B}$ is non–empty. Moreover every $a \in \mathcal{A}$ satisfies $a \leq \sup\mathcal{A}$ and every $b \in \mathcal{B}$ satisfies $b \leq \sup\mathcal{B}$, and every element c of $\mathcal{A}+2\mathcal{B}$ is of this form. Hence $c \leq \sup A + \sup B$ (*). Thus $\mathcal{A}+2\mathcal{B}$ is non–empty and bounded above, so by the Continuum Property $\sup(\mathcal{A}+2\mathcal{B})$ exists. (ii) Moreover, by (*), $\sup A + 2\sup B$ is an upper bound for $\mathcal{A}+2\mathcal{B}$. (iii) If we had $a \leq \sup\mathcal{A} - \delta$ for every $a \in \mathcal{A}$, then $\sup\mathcal{A}$ would not be the least upper bound for $\mathcal{A}$. Hence there is an element a of $\mathcal{A}$ with $a > \sup\mathcal{A} - \delta$. Likewise there is a $b \in \mathcal{B}$ with $b > \sup\mathcal{B} - \delta$. (iv) By (ii) $\sup(\mathcal{A}+2\mathcal{B}) \leq \sup\mathcal{A} + 2\sup\mathcal{B}$. We argue by contradiction. Suppose we have strict inequality. Let $\delta = \frac{1}{3}\left(\sup\mathcal{A} + 2\sup\mathcal{B} - \sup(\mathcal{A}+2\mathcal{B})\right)$. By (iii) there are $a \in \mathcal{A}$, $b \in \mathcal{B}$ such that $a+2b > \sup\mathcal{A} + 2\sup\mathcal{B} - 3\delta$. But by the definition of δ the RHS is $\sup(\mathcal{A}+2\mathcal{B})$ and this contradicts the fact that $a+2b \leq \sup(\mathcal{A}+\mathcal{B})$.

2. Let $\mathcal{A} = \left\{2 + \frac{3}{\sqrt{n}} : n \in \mathbb{N}\right\}$. Prove that $\inf\mathcal{A}$ exists and $\inf\mathcal{A} = 2$.

Proof. We have $5 = 2 + 3/\sqrt{1} \in \mathcal{A}$, so $\mathcal{A} \neq \emptyset$. We also have $2 + 3/\sqrt{n} > 2$ for every $n \in \mathbb{N}$. Thus $\mathcal{A}$ is bounded below by 2. We complete the proof by showing that there is no larger lower bound. We argue by contradiction. Suppose that $a > 2$ is a lower bound. By the Archimedean property there is an $n \in \mathbb{N}$ such that $n > 9(a-2)^{-2}$. Hence $(a-2)\sqrt{n} > 3$, $a > 2 + 3/\sqrt{n}$ contradicting the assumption on a.

3. Prove, using the definition of a limit, that $\lim_{n\to\infty} \frac{n^3+n}{n^3+n+7} = 1$.

Proof. Let $\varepsilon > 0$. Choose $N = (7/\varepsilon)^{1/3}$. Then, whenever $n > N$ we have $\left|\frac{n^3+n}{n^3+n+7} - 1\right| = \frac{7}{n^3+n+7} < \frac{7}{n^3} < \frac{7}{N^3} = \varepsilon$.

4. Suppose that $\lim_{n\to\infty} a_n = l$ and $l > 2$. Prove that there is an N such that whenever $n > N$ we have $a_n > 2$.

Proof. Let $\varepsilon = l - 2$. Choose N so that whenever $n > N$ we have $|a_n - l| < \varepsilon$. Then $-\varepsilon < a_n - l$, whence $2 = l - \varepsilon < a_n$ as required.