MATH 401, SPRING TERM 2024, MODEL SOLUTIONS TO PRACTICE EXAM 1

Note that the first exam is on Wednesday 7th February, at 1:25 in Room 011 Huck Life Sciences.

1. Differentiate with respect to x. (i) $(2x^3-1)\sin x$, (ii) $\frac{7x-4}{\ln x}$

(i) $6x^2\sin x + (2x^3-1)\cos x$. (ii) $\frac{7}{\ln x} - \frac{7x-4}{x(\ln x)^2}$.

2. The function f is defined by $f(x) = \frac{4}{x-2} - \frac{1}{x+1}$. (i) Show that $f'(x) = -\frac{3x(x+4)}{(x-2)^2(x+1)^2}$ and find the maxima and minima of f. (ii) Sketch the graph of f.

(i) $f'(x) = \frac{-4}{(x-2)^2} + \frac{1}{(x+1)^2} = \frac{-4(x+1)^2+(x-2)^2}{(x-2)^2(x+1)^2} = \frac{-3x^2-12x}{(x-2)^2(x+1)^2}$. $f'(x) = 0$ when $x = 0$ or $x = -4$. $f''(x) = \frac{8}{(x-2)^3} - \frac{2}{(x+1)^3}$ so $f''(0) = -3 < 0$ and $f''(-4) = \frac{1}{27} > 0$. Thus 0 gives a max and -4 a min. (ii) See class.

3. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be sets. Prove that $(\mathcal{A}\cap\mathcal{B})\cup\mathcal{C} = (\mathcal{A}\cup\mathcal{C})\cap(\mathcal{B}\cup\mathcal{C})$.

A	B	C	$A\cap B$	$(A\cap B)\cup C$	$A\cup C$	$B\cup C$	$(A\cup C)\cap(B\cup C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	0	1	0	0
0	1	1	0	1	1	1	1
0	1	0	0	0	0	1	0
0	0	1	0	1	1	1	1
0	0	0	0	0	0	0	0

Summary of order axioms from class (slightly different from the textbook): There is a relation "$<$" which satisfies the following axioms. a, b, c denote real numbers.

O1. Exactly one of $a < b$, $a = b$, $b < a$ holds.

O2. If $a < b$ and $b < c$, then $a < c$.

O3. If $a < b$, then $a + c < b + c$ for all c.

O4. If $a < b$ and $0 < c$, then $ac < bc$.

The expression $a > b$ means $b < a$. We also use $a \le b$ to mean "either $a < b$ or $a = b$".

4. Suppose that a, b, c are three real numbers and $0 < abc$. Prove that at least one of a, b, c is positive.

We argue by contradiction. Suppose each of a, b, c is negative or 0. If at least one of them is 0, then at once from the arithmetic axioms we have $abc = 0$ giving a contradiction. If none of a, b, c is 0 then they will all be negative, i.e. $a < 0$, $b < 0$ and $c < 0$. By Theorem 2.1 we have $0 = 0.b < a.b$. Then by Theorem 2.1 again we have $abc = ab.c < ab.0 = 0$ contradicting the hypothesis.